

# BPS States and Automorphisms

Jordi Molins and Joan Simón<sup>†</sup>

*Departament ECM, Facultat de Física, Universitat de Barcelona and Institut de Física d'Altes Energies, Diagonal 647, E-08028 Barcelona, Spain. E-mail: molins@ecm.ub.es, jsimon@ecm.ub.es.*

<sup>†</sup> *Address after October 2, 2000 : Weizmann Institute, Rehovot, Israel.*

The purpose of the present paper is twofold. In the first part, we provide an algebraic characterization of several families of  $\nu = \frac{1}{2^n}$   $n \leq 5$  BPS states in M theory, at threshold and non-threshold, by an analysis of the BPS bound derived from the  $\mathcal{N} = 1$   $D = 11$  SuperPoincaré algebra. We determine their BPS masses and their supersymmetry projection conditions, explicitly. In the second part, we develop an algebraic formulation to study the way BPS states transform under  $GL(32, \mathbb{R})$  transformations, the group of automorphisms of the corresponding SuperPoincaré algebra. We prove that all  $\nu = \frac{1}{2}$  non-threshold bound states are  $SO(32)$  related with  $\nu = \frac{1}{2}$  BPS states at threshold having the same mass. We provide further examples of this phenomena for less supersymmetric  $\nu = \frac{1}{4}, \frac{1}{8}$  non-threshold bound states.

PACS numbers: 11.10.Kk, 11.25.-w

Keywords: M-algebra, BPS states, automorphisms

## I. INTRODUCTION

The importance of BPS states in string theory has largely been emphasized in the literature. Not only they do provide a non-trivial check for the conjectured web of dualities in string theory [1] but they are also relevant for the microscopic computation of black hole entropies [2] and for studying non-perturbative phenomena in supersymmetric gauge field theories. It is therefore important to know their spectrum. One of the purposes of the present paper is to go in this direction.

There are several ways to study the existence of BPS states in string theory. Among them, one can look for classical supergravity solutions preserving some amount of supersymmetry. This has been studied extensively, giving rise to a large list of solutions involving intersections of branes and also non-threshold bound states [3,4]. One of the conclusions of all these analysis was the derivation of some rules to construct BPS supergravity solutions (harmonic superposition rule) and some techniques [5] to generate new solutions from the previously known ones by T-duality and M-reductions at angles and along boosted directions or through electro-magnetic duality.

Besides world volume solitons, there exists certainly a third possibility, which is entirely algebraic [6], and which will be the one followed in this paper. It consists in the resolution of the eigenvalue problem associated with the saturation of the BPS bound derived from the  $\mathcal{N} = 1$   $D = 11$  SuperPoincaré algebra. This approach characterizes the Clifford valued BPS states  $|\alpha\rangle$  by its mass  $\mathcal{M}$  and the amount of supersymmetry preserved ( $\nu$ ), which will generically be determined by some set of mutually commuting operators  $\{\mathcal{P}_i\}$  such that  $\mathcal{P}_i|\alpha\rangle = |\alpha\rangle \forall i$ . Both depend on the charges  $\mathcal{Z}$  carried by  $|\alpha\rangle$ . The main features of this eigenvalue problem are explained in section II. Given two single  $\nu = \frac{1}{2}$  BPS branes, their asso-

ciated supersymmetry projection operators, either commute or anticommute, giving rise to an intersection of branes (BPS state at threshold) or a truly bound state (non-threshold bound state) preserving  $\nu = \frac{1}{4}$  or  $\nu = \frac{1}{2}$  respectively. We generalize this idea by considering BPS states built up from two commuting or anticommuting BPS subsystems  $(\mathcal{M}_1, \nu_1)$  and  $(\mathcal{M}_2, \nu_2)$ . In section III, we give the mass  $\mathcal{M}(\mathcal{M}_1, \mathcal{M}_2)$  and the amount of supersymmetry  $\nu(\nu_1, \nu_2)$  preserved by the full state in both cases by proving two basic factoring theorems. Since their proof is constructive, we are also able to identify the corresponding set of projectors  $\{\mathcal{P}_i\}$  out of the ones  $\{\mathcal{P}_{1j}\}$  and  $\{\mathcal{P}_{2j}\}$  characterizing the two subsystems. This approach is clearly recursive, in the sense that any subsystem  $(\mathcal{M}_i, \nu_i)$  may be decomposed into subsubsystems  $(\mathcal{M}_{ij}, \nu_{ij})$  by application of the previously mentioned theorems (factorizable states). It is in this way that we identify and classify families of  $\nu = \frac{1}{2^n}$   $n \leq 5$  BPS states, both at threshold and non-threshold. Our analysis provides an intensive classification of the subset of BPS states in M-theory, and consequently in type IIA and type IIB theories, that we call factorizable states. We give a large list of particular configurations satisfying our conditions, but certainly not an extensive one.

All the analysis done in section III is for an arbitrary value of the central charges  $\mathcal{Z}$ 's. In section IV, we first of all comment on some of the general algebraic conditions that enhance supersymmetry through fine tuning of the central charges  $\mathcal{Z}$ 's [7]. Besides that, we give an example for the easiest set of configurations not included in our previous classification, the non-factorizable states.

The second purpose of this paper is to start analysing the way  $GL(32, \mathbb{R})$ , the group of automorphisms of the  $\mathcal{N} = 1$   $D = 11$  SuperPoincaré algebra [8,9], acts on such BPS states. We first develop a useful algebraic formulation to check that indeed  $GL(32, \mathbb{R})$  is such a group and to compute the way central charges  $\mathcal{Z}$  transform under such

finite transformations. The subgroup of  $SO(32)$  transformations is identified with that preserving the mass of BPS states. By studying some of the easiest  $SO(32)$  transformations, we prove that all non-threshold  $\nu = \frac{1}{2}$  BPS states are  $SO(32)$  related to  $\nu = \frac{1}{2}$  BPS states at threshold of the same mass. We provide some particular examples of this phenomena, which is a generalization of the well-known fact that the truly bound state  $|\alpha\rangle$  is characterized by  $\Gamma|\alpha\rangle = |\alpha\rangle$

$$\Gamma = \cos \alpha \Gamma_{01} + \sin \alpha \Gamma_{02},$$

associated with M-waves propagating in the 1-direction and 2-direction is related with an M-wave propagating in some intermediate direction  $|\alpha'\rangle$ , due to the existence of a rotation in the 12-plane  $R_{12}$  belonging to the  $SO(10)$  subgroup relating both states ( $|\alpha\rangle = R_{12}|\alpha'\rangle$ ).

Even though we do not study the orbits of  $GL(32, \mathbb{R})$ , we do provide evidence for the existence of more involved  $SO(32)$  transformations relating  $\nu = \frac{1}{4}$  and  $\nu = \frac{1}{8}$  non-threshold bound states to  $\nu = \frac{1}{4}$  and  $\nu = \frac{1}{8}$  BPS states at threshold having the same mass, thus partially generalizing the result on  $\nu = \frac{1}{2}$  non-threshold bound states. We end up with some discussion concerning open questions related to world volume realization of automorphisms.

## II. EIGENVALUE PROBLEM

The  $\mathcal{N} = 1$   $D = 11$  SuperPoincaré algebra [10,11] is basically described by

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)_{\alpha\beta} \mathcal{Z}_M + \frac{1}{2}(C\Gamma_{MN})_{\alpha\beta} \mathcal{Z}^{MN} \quad (1)$$

$$+ \frac{1}{5!}(C\Gamma_{MNPQR})_{\alpha\beta} \mathcal{Z}^{MNPQR}$$

$$[Q_\alpha, \mathcal{Z}^{M\dots}] = 0. \quad (2)$$

$Q_\alpha$  denotes the 32-component Majorana spinor generating supersymmetry, whereas  $\Gamma_{MN\dots}$  stands for the antisymmetric product of  $\Gamma_M$  matrices satisfying the Clifford algebra in eleven dimensions  $\{\Gamma_M, \Gamma_N\} = 2\eta_{MN}$ ,  $M, N = 0, 1, \dots, 9, \sharp$ . The translation operators  $P_M$  were denoted by  $\mathcal{Z}_M$ , just by notational convenience.

Since we are assuming that the above supersymmetry is valid at any energy, the spectrum of M-theory should be organized into representations of the SuperPoincaré algebra. We will be concerned with states preserving some amount of supersymmetry, thus filling in short irreducible representations of the latter algebra, the so called *BPS states*. These can be entirely characterized by purely algebraic techniques. In particular, given any M-theory state  $|\alpha\rangle$ , the positivity of the matrix  $\langle \alpha | \{Q_\alpha, Q_\beta\} | \alpha \rangle$  implies a bound on the rest mass  $\mathcal{M} = \mathcal{Z}^0$ , known as the Bogomol'ny bound. When the latter is saturated, there is a linear combination of the supersymmetry generators annihilating the state. This

means that the symmetric matrix  $\{Q_\alpha, Q_\beta\}$  has at least one zero eigenvalue ( $\det \{Q_\alpha, Q_\beta\} = 0$ ). Thus, generically, the search for such BPS states is equivalent to the resolution of the eigenvalue problem

$$\Gamma|\alpha\rangle = \mathcal{M}|\alpha\rangle, \quad (3)$$

where

$$\Gamma = \Gamma_{0m} \mathcal{Z}^m + \frac{1}{2} \Gamma_{0MN} \mathcal{Z}^{MN} + \frac{1}{5!} \Gamma_{0M_1\dots M_5} \mathcal{Z}^{M_1\dots M_5}$$

$m = 1, \dots, 9, \sharp$  and we have already used that the charged conjugation matrix may be chosen as  $C = \Gamma_0$ .

The analysis of the corresponding eigenvalue problem was solved with full generality for the  $\mathcal{N} = 1$   $D = 4$  SuperPoincaré algebra in [12], but the eleven dimensional problem is much more involved. The first outstanding comment is that all antisymmetrized products of Dirac matrices ( $\Gamma_i$ ) appearing in the operator  $\Gamma = \sum_i \mathcal{Z}^i \Gamma_i$  satisfy

$$\Gamma_i^2 = 1, \text{ tr } \Gamma_i = 0. \quad (4)$$

We will call such matrices, *single projectors*<sup>1</sup>.

Single projectors are related to  $\nu = \frac{1}{2}$  BPS states corresponding to single branes. This statement corresponds to the well-known fact that given a single brane extending along certain directions of spacetime, or equivalently, certain “central charge”  $\mathcal{Z}_1$ , eq. (3) becomes

$$\mathcal{Z}_1 \Gamma_1 |\alpha\rangle = \mathcal{M} |\alpha\rangle. \quad (5)$$

Squaring eq. (5), we derive that  $\mathcal{M} = |\mathcal{Z}_1|$  and  $\Gamma_1 |\alpha\rangle = \pm |\alpha\rangle$ , from which we infer that  $\nu = \frac{1}{2}$ , due to (4). In this way, we could rederive all  $\nu = \frac{1}{2}$  BPS states associated with single branes which are summarized in table I.

Actually, the mapping between single branes and single projectors is one to two since given a single projector  $\Gamma_i$ , there exists a second one  $\tilde{\Gamma}_i$  such that  $\Gamma_i \tilde{\Gamma}_i = \mathbb{I}$ , since  $\Gamma_0 \Gamma_1 \dots \Gamma_\sharp = \mathbb{I}$ . This observation agrees perfectly with the known fact that the electrical charges  $\mathcal{Z}^{0m}$  and  $\mathcal{Z}^{0m_1\dots m_4}$  correspond to M9-brane and Mkk, respectively, since  $\tilde{\Gamma}_1 = \Gamma_{0m_1\dots m_9}$  and  $\tilde{\Gamma}_2 = \Gamma_{0m_1\dots m_6}$ . From now on we will always be assuming that all single projectors  $\{\Gamma_i\}$  appearing in our computations satisfy  $\Gamma_i \Gamma_j \neq \mathbb{I} \forall i, j$   $i \neq j$ .

The second basic comment concerns the commutativity or anticommutativity among two single projectors. That is, given any two single projectors  $\Gamma_i, \Gamma_j$ , they either commute  $[\Gamma_i, \Gamma_j] = 0$  or anticommute  $\{\Gamma_i, \Gamma_j\} = 0$ . Let us consider the commuting case

$$\Gamma = \mathcal{Z}_1 \Gamma_1 + \mathcal{Z}_2 \Gamma_2, \quad [\Gamma_1, \Gamma_2] = 0. \quad (6)$$

<sup>1</sup>Even though they are not projectors in a strict sense.

We obtain the new eigenvalue problem

$$\Gamma'|\alpha\rangle = 2\mathcal{Z}_1\mathcal{Z}_2\Gamma_1\Gamma_2|\alpha\rangle = \mathcal{M}'|\alpha\rangle, \quad (7)$$

where  $\mathcal{M}' = \mathcal{M}^2 - (\mathcal{Z}_1)^2 + (\mathcal{Z}_2)^2$ , by squaring eq. (3). Squaring eq. (7) fixes

$$\mathcal{M}' = \pm 2\mathcal{Z}_1\mathcal{Z}_2 \quad (8)$$

$$\Gamma_1\Gamma_2|\alpha\rangle = \pm|\alpha\rangle, \quad (9)$$

that determines, upon substitution into the original eigenvalue problem, the BPS mass

$$\mathcal{M} = |\mathcal{Z}_1 \pm \mathcal{Z}_2| \quad (10)$$

and the supersymmetry projection conditions

$$\Gamma_1|\alpha\rangle = \Gamma_2|\alpha\rangle = \pm|\alpha\rangle. \quad (11)$$

The above BPS state preserves  $\nu = \frac{1}{4}$  since  $[\Gamma_1, \Gamma_2] = 0$  and  $\text{tr}(\Gamma_1\Gamma_2) = 0$ , by hypothesis. Of course, we could have derived the same conclusions by using an explicit representation of the  $\Gamma$  matrices in a basis where both,  $\Gamma_1$  and  $\Gamma_2$  were diagonal. We summarize all possible  $\nu = \frac{1}{4}$  BPS configurations formed by two commuting single branes in table II. We use the notation  $Mp \perp Mq(n)$ , where  $p$  and  $q$  indicate the space dimensions along which the branes are extended, whereas  $n$  stands for the common space dimensions.

On the other hand, in the anticommuting case, when squaring eq. (3), we derive that the BPS mass is given by

$$\mathcal{M} = \sqrt{(\mathcal{Z}_1)^2 + (\mathcal{Z}_2)^2}, \quad (12)$$

whereas the projection condition becomes

$$\frac{\mathcal{Z}_1\Gamma_1 + \mathcal{Z}_2\Gamma_2}{\sqrt{(\mathcal{Z}_1)^2 + (\mathcal{Z}_2)^2}}|\alpha\rangle = \pm|\alpha\rangle. \quad (13)$$

It should be stressed that the coefficients appearing in the above condition parametrize a circle of unit radius ( $\cos\alpha = \mathcal{Z}_1/\sqrt{(\mathcal{Z}_1)^2 + (\mathcal{Z}_2)^2}$ ). The previous BPS state is  $\nu = \frac{1}{2}$  due to the appearance of a unique projection condition. It corresponds to a non-threshold bound state, since  $\mathcal{M} \leq \mathcal{Z}_1 + \mathcal{Z}_2$ . Alternatively, we could have worked in a basis where

$$\begin{aligned} \Gamma_1 &= \mathbb{I}_{16} \otimes \tau_3 \\ \Gamma_2 &= \mathbb{I}_{16} \otimes \tau_1. \end{aligned} \quad (14)$$

In this way, the condition of vanishing determinant becomes

$$\left[ \det \begin{pmatrix} \mathcal{Z}_1 - \mathcal{M} & \mathcal{Z}_2 \\ \mathcal{Z}_2 & -(\mathcal{Z}_1 + \mathcal{M}) \end{pmatrix} \right]^{16} = [\mathcal{M}^2 - ((\mathcal{Z}_1)^2 + (\mathcal{Z}_2)^2)]^{16} = 0, \quad (15)$$

from which we obtain the same conclusions as in the previous analysis. As before, we summarize all possible non-threshold bound states that one can construct out of two single branes in table III.

Up to now, we have solved eq. (3) whenever  $\Gamma$  contains one or two single projectors. One could continue in this direction classifying the set of BPS states in terms of the number of single branes/projectors ( $N$ ) involved in the eigenvalue problem (3). Actually, once  $N$  is fixed,

there are  $\binom{N}{2}$  pairs of single projectors, giving rise to  $\binom{N}{2} + 1$  inequivalent configurations uncovering the set of all inequivalent commutation relations among  $N$  single projectors. It is interesting to remark that one can map the problem of classifying the set of initial  $\Gamma$  operators to the problem of classifying the inequivalent graphs of  $N$  points linked by  $L$  lines<sup>2</sup>. One can draw a point for every single projector involved in  $\Gamma$ , and link any two of them whenever the corresponding single projectors anticommute, leaving them unlinked otherwise.

To solve the eigenvalue problem (3) one can apply an algorithm already used in [13] and in the previous discussion. It consists in acting on the left of eq. (3) with  $\Gamma$ , giving rise to  $\Gamma^2|\alpha\rangle = \mathcal{M}^2|\alpha\rangle$ , which is equivalent to

$$\left[ \sum_{i=1}^N (\mathcal{Z}_i)^2 + \sum_{i<j}^N \mathcal{Z}_i\mathcal{Z}_j \{\Gamma_i, \Gamma_j\} \right] |\alpha\rangle = \mathcal{M}^2|\alpha\rangle.$$

Thus, whenever the set of  $N$  single projectors  $\{\Gamma_i\}$  anticommute  $\{\Gamma_i, \Gamma_j\} = 0 \forall i, j$ , the above equation already fixes the eigenvalue to be

$$\mathcal{M}^2 = \sum_{i=0}^N (\mathcal{Z}_i)^2,$$

which when put it back into eq. (3), gives rise to the projection condition

$$\sum_i \frac{\mathcal{Z}_i}{\mathcal{M}} \Gamma_i |\alpha\rangle = \pm |\alpha\rangle, \quad (16)$$

which corresponds to a non-threshold bound state preserving  $\nu = \frac{1}{2}$  build up of  $N$  single projectors.

If there exist some pairs of commuting single projectors  $\{\Gamma_i, \Gamma_j\} = 2\Gamma_i\Gamma_j$ , one can map equation  $\Gamma^2|\alpha\rangle = \mathcal{M}^2|\alpha\rangle$  into a new eigenvalue problem

$$\Gamma'|\alpha\rangle = \mathcal{M}'|\alpha\rangle, \quad (17)$$

---

<sup>2</sup>JS would like to thank Ignasi Mundet for pointing out such a connection.

where

$$\Gamma' = 2 \sum_{i < j}^N \mathcal{Z}_i \mathcal{Z}_j \Gamma_i \Gamma_j,$$

and

$$\mathcal{M}' = \mathcal{M}^2 - \sum_{i=1}^N (\mathcal{Z}_i)^2.$$

Notice that  $[\Gamma, \Gamma'] = 0$  since  $\Gamma' = \Gamma^2 - (\mathcal{M}' - \mathcal{M}^2)\mathbb{I}$ . Given the new eigenvalue problem (17), one can proceed as for the original one (3). This defines an algorithm that will typically finish whenever  $\Gamma^{(p)}$ , the operator appearing in the  $p$ -th eigenvalue problem, consists of a linear combination of anticommuting single projectors<sup>3</sup>.

Whenever the algorithm is solved,  $\mathcal{M}^{(p)}$  is fixed, and by construction, the set of all  $\mathcal{M}^{(i)}$   $i = 0, \dots, p$  becomes fixed. Inserting these eigenvalues back into the defining equations  $\Gamma^{(i)}|\alpha\rangle = \mathcal{M}^{(i)}|\alpha\rangle$ , gives rise to a set of  $(p+1)$  commuting projection conditions

$$\mathcal{P}^{(i)}|\alpha\rangle = |\alpha\rangle, \quad (18)$$

determining the amount of preserved supersymmetry to be  $\nu = \frac{1}{2^{p+1}}$ .

Some important comments are in order at this point. First of all, that  $\nu = \frac{1}{2^{p+1}}$  relies on the fact that  $\text{tr}(\Gamma^{(r)}\Gamma^{(s)}) = 0$  for  $r \neq s$  and  $r, s \leq p$ . Furthermore, it is assumed that no  $\mathcal{P}^{(i)}$  can be written as the product of other projectors. If this was the case, there would be an enhancement of supersymmetry. The easiest example of this phenomena is the well-known fact that one can add single branes for free in certain configurations of intersecting branes.

We would also like to stress that whenever  $p > 4$ , the whole set of 32 supercharges are already broken and the corresponding state does not belong to any supersymmetry multiplet. For later convenience, it is worthwhile to keep in mind that the above statements apply for arbitrary values of the charges  $\mathcal{Z}_i$ . There might be enhancement of supersymmetry for particular values of the whole set of  $\{\mathcal{Z}_i\}$  under consideration.

### III. FACTORIZABLE STATES

It is the aim of this section to describe some families of BPS states solving the eigenvalue problem (3). Before

---

<sup>3</sup>They are always linear combinations of single projectors due to  $\Gamma_0\Gamma_1 \dots \Gamma_9\Gamma_{10} = 1$ . For instance, from the definition of  $\Gamma'$ , it is clear that  $\Gamma_i\Gamma_j$  are single projectors since  $(\Gamma_i\Gamma_j)^2 = 1$  and  $\text{tr}(\Gamma_i\Gamma_j) = 0$ , by hypothesis. Analogous arguments apply for  $\Gamma^{(p)}$  operators.

defining precisely these families, we would like to present two *factoring theorems* that generalize previous results on two single brane configurations and allow us to divide our original eigenvalue problem into simpler ones. Let us consider the case in which the initial  $\Gamma$  operator decomposes according to

$$\Gamma := \sum_a \mathcal{Z}_a \Gamma_a + \sum_i \mathcal{Z}_i \Gamma_i =: \Gamma_{[a]} + \Gamma_{[i]}, \quad (19)$$

where  $a$  and  $i$  label the first and second subsystem, respectively. We will assume that both subsystems do have a solution of their associated eigenvalue problems, i.e.

$$\Gamma_{[a]}|\alpha\rangle = \mathcal{M}_{[a]}|\alpha\rangle \quad (20)$$

$$\Gamma_{[i]}|\alpha\rangle = \mathcal{M}_{[i]}|\alpha\rangle. \quad (21)$$

Then, if  $[\Gamma_a, \Gamma_i] = 0 \ \forall a, i$ , the eigenvalue problem is solved by

$$\mathcal{M} = |\mathcal{M}_{[a]} \pm \mathcal{M}_{[i]}| \quad (22)$$

$$\nu = \nu_{[a]} \cdot \nu_{[i]}, \quad (23)$$

whereas if  $\{\Gamma_a, \Gamma_i\} = 0 \ \forall a, i$ , the solution is given by

$$\mathcal{M} = \sqrt{(\mathcal{M}_{[a]})^2 + (\mathcal{M}_{[i]})^2} \quad (24)$$

$$\nu = \frac{1}{2} \cdot (2\nu_{[a]}) \cdot (2\nu_{[i]}). \quad (25)$$

*Proof*

(a) If  $[\Gamma_a, \Gamma_i] = 0 \ \forall a, i \Rightarrow [\Gamma_{[a]}, \Gamma_{[i]}] = 0$ . This means that we can simultaneously diagonalize  $\Gamma_{[a]}$  and  $\Gamma_{[i]}$  with eigenvalues  $\pm\mathcal{M}_{[a]}$ ,  $\pm\mathcal{M}_{[i]}$ , respectively. Thus  $\mathcal{M} = |\mathcal{M}_{[a]} \pm \mathcal{M}_{[i]}|$ , whereas the amount of preserved supersymmetry is  $\nu = \nu_{[a]} \cdot \nu_{[i]}$ . Actually, the set of projectors  $\{\mathcal{P}\}$  satisfying  $\mathcal{P}|\alpha\rangle = +|\alpha\rangle^4$ , is given by

$$\{\mathcal{P}\} = \{\mathcal{P}_{[a]}, \mathcal{P}_{[i]}\}. \quad (26)$$

(b) If  $\{\Gamma_a, \Gamma_i\} = 0 \ \forall a, i \Rightarrow \Gamma^2 = \Gamma_{[a]}^2 + \Gamma_{[i]}^2$ . But,  $\Gamma_{[a]}^2 = \sum_a (\mathcal{Z}_a)^2 + \Gamma'_{[a]}$  and  $\Gamma_{[i]}^2 = \sum_i (\mathcal{Z}_i)^2 + \Gamma'_{[i]}$ . Thus the original eigenvalue problem is mapped after the first step of the algorithm to

$$(\Gamma'_{[a]} + \Gamma'_{[i]})|\alpha\rangle = \mathcal{M}'|\alpha\rangle, \quad (27)$$

where  $\mathcal{M}' = \mathcal{M}^2 - \sum_a (\mathcal{Z}_a)^2 - \sum_i (\mathcal{Z}_i)^2$ . The important point is that  $[\Gamma'_{[a]}, \Gamma'_{[i]}] = 0 \ \forall a, i$  since  $\Gamma'_a = 2\mathcal{Z}_b\mathcal{Z}_c\Gamma_b\Gamma_c$  and  $\Gamma'_i = 2\mathcal{Z}_k\mathcal{Z}_l\Gamma_k\Gamma_l$ . We can then apply the previous result (a) to fix  $\mathcal{M}' = \mathcal{M}'_{[a]} + \mathcal{M}'_{[i]}$ , from which we derive that

---

<sup>4</sup>From now on, we will restrict ourselves to the plus sign projection keeping in mind that both of them ( $\pm$ ) are still possible

$$\mathcal{M} = \sqrt{\mathcal{M}_{[a]}^2 + \mathcal{M}_{[i]}^2}, \quad (28)$$

since  $\mathcal{M}_{[a]}^2 = \mathcal{M}'_{[a]} + \sum_a (\mathcal{Z}_a)^2$  and  $\mathcal{M}_{[i]}^2 = \mathcal{M}'_{[i]} + \sum_i (\mathcal{Z}_i)^2$ , by hypothesis. Furthermore,  $\nu' = \nu'_{[a]} \nu'_{[i]}$  by (a). By construction,  $\nu'_{[a]} = 2\nu_{[a]}$  and  $\nu'_{[i]} = 2\nu_{[i]}$ . Finally, the amount of preserved supersymmetry of the original problem is  $\nu = \frac{1}{2}\nu'$ , since one must take into account the original projection condition  $\Gamma|\alpha\rangle = \mathcal{M}|\alpha\rangle$ . We conclude that  $\nu = 2\nu_{[a]} \cdot \nu_{[i]}$  as we claimed. Actually,

$$\{\mathcal{P}\} = \{\mathcal{P}_0, \mathcal{P}'_{[a]}, \mathcal{P}'_{[i]}\}, \quad (29)$$

where  $\mathcal{P}_0$  stands for the original projector and  $\mathcal{P}'_{[a]}, \mathcal{P}'_{[i]}$  stand for the projector conditions associated with  $\Gamma'_{[a]}, \Gamma'_{[i]}$ , respectively.

There is a nice geometrical picture emerging from our forementioned connection to graph theory. Those BPS states satisfying the conditions for the first factoring theorem correspond to disconnected graphs, whereas those associated with the second theorem correspond to graphs containing subgraphs ( $\mathcal{G}_i$ ) such that all  $N_i$  points belonging to  $\mathcal{G}_i$  are linked with all  $N_j$  points in  $\mathcal{G}_j$   $i \neq j$ . It would be interesting to use known results from graph theory to classify the full set of BPS states in M-theory, and viceversa, to use information from branes into graph theory.

In the above proofs, we assumed the existence of a non-trivial solution to  $\Gamma_{[a]}|\alpha\rangle = \mathcal{M}_{[a]}|\alpha\rangle$  and  $\Gamma_{[i]}|\alpha\rangle = \mathcal{M}_{[i]}|\alpha\rangle$ , so in this respect, they are completely general. In particular, we claim that whenever  $\Gamma_{[a]}$  and  $\Gamma_{[i]}$  admit a similar decomposition into commuting/anticommuting subsystems such that we can again apply the factoring theorems, the original eigenvalue problem (3) has a solution. By construction, all BPS states captured in this way will preserve  $\nu = \frac{1}{2^{p+1}} p \leq 4$ . We shall now describe, in more detail, the forementioned BPS states, according to the value of  $p$ .

#### A. $\nu = \frac{1}{2}$ BPS states

It should be clear from our previous discussions that the most general  $\nu = \frac{1}{2}$  BPS state described by our hypothesis is given by a linear combination of anticommuting single projectors. Actually, its mass is given by

$$\mathcal{M}_{1/2} = \sqrt{\sum_{i=1}^N (\mathcal{Z}_i)^2},$$

whereas the projection condition is

$$\mathcal{P}|\alpha\rangle = |\alpha\rangle, \quad \mathcal{P} = \frac{1}{\mathcal{M}_{1/2}} \sum_{i=1}^N \mathcal{Z}_i \Gamma_i.$$

Equivalently, the coefficients in the linear combination  $\mathcal{P} = \sum_i a^i \Gamma_i$  parametrize an  $S^{N-1}$  sphere, so that they

can always be rewritten in terms of trigonometric functions.

Notice that these states depend on the number of single branes ( $N$ ) forming them. For  $N = 1$ , we recover the usual single branes (see table I), whereas for  $N \geq 2$ , we find families of non-threshold bound states, since their mass satisfies

$$\mathcal{M}_{1/2} \leq \sum_{i=1}^N \mathcal{M}_{1/2}^{(i)}.$$

All  $N = 2$  possibilities have already been summarized in table III. The latter are the building blocks for higher  $N$  non-threshold bound states. Generically, to construct a  $(\nu = \frac{1}{2}, N+1)$  non-threshold bound state one must just look for a single projector anticommuting with all  $N$  single projectors characterizing the original  $(\nu = \frac{1}{2}, N)$  bound state. Again, all this information is already encoded in table III. Thus from  $M2 \perp M2(1)$ , one can add a third M2-brane giving rise to two inequivalent arrays, either

$$\begin{array}{l} M2 : 1 \ 2 \ - \ - \ - \ - \ - \ - \\ M2 : \ - \ 2 \ 3 \ - \ - \ - \ - \ - \\ M2 : 1 \ - \ 3 \ - \ - \ - \ - \ - \end{array} \quad (30)$$

or

$$\begin{array}{l} M2 : 1 \ 2 \ - \ - \ - \ - \ - \ - \\ M2 : 1 \ - \ 3 \ - \ - \ - \ - \ - \\ M2 : 1 \ - \ - \ 4 \ - \ - \ - \ - \ - \end{array}, \quad (31)$$

which from now on will be denoted by  $2^3\{0, 3, 0\}$  and  $2^3\{3, 0, 1\}$ , respectively, using the same notation as in [4]. Generically  $p_1^{r_1} \dots p_s^{r_s} \{n_1, \dots, n_N\}$  will denote a configuration of  $N = r_1 + \dots + r_s$  branes of  $p_1, \dots, p_s$  world space dimensions such that the number of columns in the corresponding array with  $i$  common workspace directions is  $n_i$   $i \in \{1, \dots, N\}$ . Clearly, configuration (31) can be extended to  $2^N\{N, 0, \dots, 1\}$   $1 \leq N \leq 9$ .

There is a large number of states belonging to this category, which we do not have the intention to classify extensively, the whole set being defined in an intensive way by  $\mathcal{P}|\alpha\rangle = |\alpha\rangle$ . Some particular examples are given by  $2^2 5^1 \{2, 2, 1\}$ ,  $2^1 5^2 \{4, 4, 0\}$ ,  $2^1 5^1 6^1 \{6, 2, 1\}$ ,  $6^3 \{4, 2, 3\}$ ,  $6^2 9^1 \{0, 6, 3\}$ ,  $2^2 5^1 6^1 \{6, 0, 3, 0\}$ ,  $2^1 6^1 9^2 \{2, 2, 6, 1\}$ ,  $\dots$

#### B. $\nu = \frac{1}{4}$ BPS states

There exist two possibilities to build up  $\nu = \frac{1}{4}$  BPS states from our factoring theorems. One can either apply the first one giving rise to BPS masses

$$\mathcal{M}_{1/4}^{(1)} = \mathcal{M}_{1/2} + \mathcal{M}_{1/2}, \quad (32)$$

or the second one, giving rise to

$$\mathcal{M}_{1/4}^{(2)} = \sqrt{(\mathcal{M}_{1/2})^2 + (\mathcal{M}_{1/4}^{(1)})^2}. \quad (33)$$

In the first case, the projection conditions are given by the union of the subsystem projections (26)

$$\mathcal{P}_1|\alpha\rangle = \mathcal{P}_2|\alpha\rangle = |\alpha\rangle. \quad (34)$$

In the second case, it can be seen by using the second factoring theorem (29) and applying the previously described algorithm that

$$\begin{aligned} \mathcal{P}_2\mathcal{P}_3|\alpha\rangle &= |\alpha\rangle \\ \mathcal{P}_0|\alpha\rangle &= |\alpha\rangle, \end{aligned} \quad (35)$$

where  $\mathcal{P}_0 = \frac{1}{\mathcal{M}_{1/4}^{(2)}} \left\{ \mathcal{P}_1\mathcal{M}_{1/2} + \mathcal{P}_2\mathcal{M}_{1/4}^{(1)} \right\}$ .

Equations (32) and (34) describe a threshold bound state (intersection), whose components are the previously discussed  $\nu = \frac{1}{2}$  BPS states. As such, they will be characterized by two integer positive numbers  $(N_1, N_2)$  describing the number of single branes in both subsystems. The particular case  $N_1 = N_2 = 1$  corresponds to the standard intersection of two branes (see table II). Now we see there are more general configurations involving non-threshold bound states in both subsystems. As before, all needed information is already contained in tables II and III. For example, for  $N_1 = 1$  and  $N_2 = 2$  one can find, among many others

$$\begin{aligned} &2^25^1\{3, 3, 0\} \\ &5^3\{6, 3, 1\} \\ &2^3\{4, 1, 0\} \\ &2^16^2\{6, 1, 2\}, \end{aligned} \quad (36)$$

when  $N_1 = N_2 = 2$ ,

$$\begin{aligned} &2^25^2\{2, 3, 2, 0\} \\ &5^4\{4, 2, 4, 0\} \\ &2^26^2\{6, 0, 2, 1\} \\ &2^4\{4, 2, 0, 0\} \dots \end{aligned} \quad (37)$$

It is straightforward to increase the values of  $N_1, N_2$  by iteration of the latter process.

On the other hand, equations (33) and (35) describe non-threshold bound states built from  $\nu = \frac{1}{2}$  BPS states and the above  $\nu = \frac{1}{4}$  ones. As such, they will depend on three integer numbers  $(N_i)$ . Let us comment on the easiest examples. When  $N_1 = N_2 = N_3 = 1$ , take any configuration in table II and look for a third single brane whose associated single projector anticommutes with the latter two. For example, one can derive  $2^25^1\{4, 1, 1\}$  from  $M2 \perp M5(1)$ ,  $5^26^1\{2, 7, 0\}$  from  $Mkk \perp M5(3)$  or  $6^3\{4, 4, 2, 0\}$  from  $Mkk \perp Mkk(2)$ .

When  $N_1 = 1, N_2 = 2, N_3 = 1$ , take any configuration from  $\mathcal{M}_{1/4}^{(1)}$  with  $N_1 = 1, N_2 = 2$  and look for a fourth

single projector anticommuting with the latter three. In particular, from the examples written down in (36), one can derive the existence of

$$\begin{aligned} &2^35^1\{3, 1, 2, 0\} \\ &5^4\{5, 1, 3, 1\} \\ &2^4\{3, 1, 1, 0\} \\ &2^26^2\{7, 1, 1, 1\}, \end{aligned} \quad (38)$$

respectively. When  $N_1 = N_2 = 2, N_3 = 1$  proceed as before, but starting from  $\mathcal{M}_{1/4}^{(1)}$  with  $N_1 = N_2 = 2$ . One can immediately find

$$\begin{aligned} &2^35^2\{2, 3, 0, 2, 0\} \\ &2^15^4\{4, 2, 2, 2, 0\} \\ &2^25^16^2\{4, 2, 0, 2, 1\} \\ &2^45^1\{4, 3, 1, 0, 0\} \dots \end{aligned} \quad (39)$$

from those quoted in (37).

It should be clear that to increase  $N'_3 = N_3 + j$  ( $N_1, N_2$  fixed), one should look for  $j$  single projectors anticommuting with all projectors describing the  $(N_1, N_2, N_3)$  configuration and among themselves. On the other hand, for a fixed  $N_3$ , one should look for the subset of all  $\mathcal{M}_{1/4}^{(1)}$  configurations whose  $(N_1 + N_2)$  single projectors anticommute with all  $N_3$  single projectors of the third subsystem.

### C. $\nu = \frac{1}{8}$ BPS states

Let us consider the different BPS states we can build from the first factoring theorem. In this case,  $\nu = \frac{1}{8} = \nu_1 \cdot \nu_2 \Rightarrow \nu_1 = \frac{1}{2}, \nu_2 = \frac{1}{4}$  (or viceversa), and we already know there are two different ways to get  $\nu_2 = \frac{1}{4}$  BPS states. We are thus led to consider the BPS masses

$$\mathcal{M}_{1/8}^{(1)} = \mathcal{M}_{1/2} + \mathcal{M}_{1/4}^{(1)} \quad (40)$$

$$\mathcal{M}_{1/8}^{(2)} = \mathcal{M}_{1/2} + \mathcal{M}_{1/4}^{(2)}, \quad (41)$$

with associated projection conditions

$$\mathcal{P}_1|\alpha\rangle = \mathcal{P}_2|\alpha\rangle = \mathcal{P}_3|\alpha\rangle = |\alpha\rangle \quad (42)$$

$$\mathcal{P}_1|\alpha\rangle = \mathcal{P}_3\mathcal{P}_4|\alpha\rangle = \tilde{\mathcal{P}}_2|\alpha\rangle = |\alpha\rangle, \quad (43)$$

corresponding to the joining of the subsystem projections, where  $\tilde{\mathcal{P}}_2 = \frac{1}{\mathcal{M}_{1/4}^{(2)}} \left\{ \mathcal{P}_2\mathcal{M}_{1/2} + \mathcal{P}_3\mathcal{M}_{1/4}^{(1)} \right\}$ .

Equations (40) and (42) describe  $\nu = \frac{1}{8}$  BPS states at threshold depending on three positive integers  $(N_i)$ . The particular case  $N_i = 1$   $i = 1, 2, 3$  is the well-known one involving triple intersections of single branes. They are obtained from table III by looking for a third single projector commuting with previous two. Examples involving Mkk-monopoles or M9-branes are

$$\begin{aligned}
&2^1 5^1 6^1 \{6, 2, 1\}, \\
&5^1 6^2 \{6, 1, 3\}, \\
&2^1 6^1 9^1 \{4, 5, 1\}, \\
&\dots
\end{aligned}$$

The search of  $N_1 = 1$  states is equivalent to looking for  $\Gamma_1$  single projectors commuting with the projectors characterizing the  $\mathcal{M}_{1/4}^{(1)}$  subsystem ( $N_2, N_3$ ). In this way, by adding adequate single branes one can derive the existence of

$$\begin{aligned}
&2^3 5^1 \{3, 4, 0, 0\} \\
&2^1 5^3 \{4, 3, 2, 0\} \\
&2^4 \{6, 1, 0, 0\} \\
&2^1 5^1 6^2 \{2, 5, 1, 1\},
\end{aligned}$$

from the previously discussed  $N_2 = 1$   $N_3 = 2$   $\mathcal{M}_{1/4}^{(1)}$  states. In exactly the same way, from the  $N_2 = N_3 = 2$   $\mathcal{M}_{1/4}^{(1)}$  states mentioned before we obtain,

$$\begin{aligned}
&2^3 5^2 \{3, 2, 3, 0, 0\}, \\
&2^1 5^4 \{3, 3, 3, 1, 0\}, \\
&2^3 6^2 \{4, 2, 2, 1, 0\}, \\
&2^5 \{6, 2, 0, 0, 0\}.
\end{aligned}$$

To increase the value of  $N_1$ , keeping  $N_2, N_3$  fixed, we must look for  $\Gamma_i$  single projectors commuting with projectors characterizing  $\mathcal{M}_{1/4}^{(1)}$  and anticommuting with  $\Gamma_1$ . We give some particular examples for the first non-trivial example,  $N_i = 2$   $i = 1, 2, 3$

$$\begin{aligned}
&2^4 5^2 \{4, 2, 2, 1, 0, 0\}, \\
&2^2 5^4 \{2, 4, 3, 0, 1, 0\}, \\
&2^4 6^2 \{3, 2, 3, 1, 0, 0\}, \\
&2^6 \{6, 3, 0, 0, 0, 0\}.
\end{aligned}$$

Equations (41) and (43) describe  $\nu = \frac{1}{8}$  BPS states at threshold built from  $\mathcal{M}_{1/2}$  and  $\mathcal{M}_{1/4}^{(2)}$ . As such, they depend on four integer numbers. Generically, to construct such states one should start from  $\mathcal{M}_{1/4}^{(2)}$  and look for a projector  $\mathcal{P}_1$  commuting with  $\mathcal{P}_3 \mathcal{P}_4$  and  $\tilde{\mathcal{P}}_2$ . We shall provide some particular examples for the lower integer values. For instance, when  $N_i = 1$   $i = 1, 2, 3, 4$  one may construct

$$\begin{aligned}
&2^3 5^1 \{4, 2, 1, 0\}, \\
&5^3 6^1 \{1, 5, 3, 0\}, \\
&2^1 6^3 \{4, 2, 4, 0\}, \\
&2^4 \{4, 2, 0, 0\};
\end{aligned}$$

when  $N_i = 1$   $i = 1, 2, 3$  and  $N_4 = 2$ ,

$$\begin{aligned}
&2^4 5^1 \{3, 2, 2, 0, 0\}, \\
&2^5 \{5, 1, 1, 0, 0\}, \\
&2^3 6^2 \{5, 3, 1, 1, 0\}, \\
&5^5 \{3, 3, 1, 2, 1\};
\end{aligned}$$

whereas for  $N_i = 1$   $N_j = 2$   $i = 1, 2$  and  $j = 3, 4$  we find, among others

$$\begin{aligned}
&2^4 5^2 \{3, 2, 1, 2, 0, 0\}, \\
&2^1 5^5 \{3, 2, 1, 3, 1, 0\}, \\
&2^5 5^1 \{4, 4, 1, 0, 0, 0\}, \\
&2^3 5^1 6^2 \{3, 2, 1, 2, 1, 0\}.
\end{aligned}$$

On the other hand, if we intend to apply the second factoring theorem, then  $\nu = \frac{1}{8} = \frac{1}{2} 2\nu_1 \cdot 2\nu_2$  admits as solutions either  $\nu_1 = \nu_2 = \frac{1}{4}$  or  $\nu_1 = \frac{1}{2}$ ,  $\nu_2 = \frac{1}{8}$ . We are thus led to consider eight more families of BPS states,

$$\mathcal{M}_{1/8}^{(i+j+1)} = \sqrt{\left(\mathcal{M}_{1/4}^{(i)}\right)^2 + \left(\mathcal{M}_{1/4}^{(j)}\right)^2} \quad i, j = 1, 2 \quad (44)$$

$$\mathcal{M}_{1/8}^{(a+5)} = \sqrt{\mathcal{M}_{1/2}^2 + \left(\mathcal{M}_{1/8}^{(a)}\right)^2} \quad a = 1, \dots, 5 \quad (45)$$

which correspond to non-threshold bound states preserving  $\nu = \frac{1}{8}$ . Let us analyze the inequivalent configurations described by (44) and (45), in particular, their projection conditions.

$\mathcal{M}_{1/8}^{(3)}$  describes a non-threshold bound state built from two  $\mathcal{M}_{1/4}^{(1)}$  subsystems. It will thus be characterized by four integers. As such, its projector conditions are given by

$$\begin{aligned}
&\mathcal{P}_1 \mathcal{P}_2 |\alpha\rangle = |\alpha\rangle \\
&\mathcal{P}_3 \mathcal{P}_4 |\alpha\rangle = |\alpha\rangle,
\end{aligned} \quad (46)$$

while its non-threshold nature is due to

$$\mathcal{P}_0 |\alpha\rangle = |\alpha\rangle, \quad (47)$$

where  $\mathcal{P}_0 = \frac{1}{\mathcal{M}_{1/8}^{(3)}} \left\{ \mathcal{P}_1 \mathcal{M}_{1/4}^{(1)} + \mathcal{P}_3 \mathcal{M}_{1/4}^{(1)} \right\}$ . Particular examples with  $N_i = 1$   $i = 1, 2, 3, 4$  are

$$\begin{aligned}
&2^3 5^1 \{4, 2, 1, 0\}, \\
&2^2 5^1 6^1 \{4, 2, 1, 1\}, \\
&5^3 6^1 \{2, 5, 3, 0\}.
\end{aligned} \quad (48)$$

The general prescription would be to start from an  $\mathcal{M}_{1/4}^{(2)}$  configuration ( $N_1, N_2, N_3$ ) and to look for a single projector commuting with  $N_1$  single projectors and anticommuting with  $N_2 + N_3$  ones. In this way, from configurations (38) one may think of

$$\begin{aligned}
&2^3 5^2 \{3, 3, 1, 1, 0\}, \\
&5^4 6^1 \{2, 4, 1, 2, 1\}, \\
&2^4 5^1 \{5, 1, 2, 0, 0\}, \\
&2^2 5^1 6^1 \{5, 2, 1, 1, 1\},
\end{aligned}$$

respectively.

Analogously,  $\mathcal{M}_{1/8}^{(4)}$  describes a non-threshold bound state built from two anticommuting  $\mathcal{M}_{1/4}^{(1)}$  and  $\mathcal{M}_{1/4}^{(2)}$  subsystems being characterized by five integers. That one has  $\mathcal{M}_{1/4}^{(1)}$  is again described by

$$\mathcal{P}_1 \mathcal{P}_2 |\alpha\rangle = |\alpha\rangle, \quad (49)$$

whereas  $\mathcal{M}_{1/4}^{(2)}$ , which contains three subsystems  $\{\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5\}$ , is described by

$$\mathcal{P}_4 \mathcal{P}_5 |\alpha\rangle = |\alpha\rangle, \quad (50)$$

following the discussion on the second factoring theorem. The non-threshold character of the state is inherited from

$$\mathcal{P}_0 |\alpha\rangle = |\alpha\rangle, \quad (51)$$

where  $\mathcal{P}_0 = \frac{1}{\mathcal{M}_{1/8}^{(4)}} \left\{ \mathcal{P}_1 \mathcal{M}_{1/4}^{(1)} + \mathcal{P}_3 \mathcal{M}_{1/2} + \mathcal{P}_4 \mathcal{M}_{1/4}^{(1)} \right\}$ . The general prescription to build up such states would be to start from an  $\mathcal{M}_{1/8}^{(3)}$  configuration  $(N_1, \dots, N_4)$  and to look for a single projector anticommuting with all  $N_1 + \dots + N_4$  previous ones. To illustrate this point, one can check that the following configurations can be derived from (48)

$$\begin{aligned} &2^3 5^2 \{4, 1, 2, 1, 0\}, \\ &2^3 5^1 6^1 \{3, 3, 1, 0, 1\}, \\ &5^3 6^2 \{1, 3, 4, 2, 0\}. \end{aligned} \quad (52)$$

Similarly, BPS states  $\mathcal{M}_{1/8}^{(5)}$  depending on six integers are characterized by

$$\begin{aligned} \mathcal{P}_2 \mathcal{P}_3 |\alpha\rangle &= |\alpha\rangle \\ \mathcal{P}_5 \mathcal{P}_6 |\alpha\rangle &= |\alpha\rangle \\ \mathcal{P}_0 |\alpha\rangle &= |\alpha\rangle \end{aligned} \quad (53)$$

where

$$\mathcal{P}_0 = \frac{1}{\mathcal{M}_{1/8}^{(5)}} \left\{ \mathcal{P}_1 \mathcal{M}_{1/2} + \mathcal{P}_2 \mathcal{M}_{1/4}^{(1)} + \mathcal{P}_4 \mathcal{M}_{1/2}' + \mathcal{P}_5 \mathcal{M}_{1/4}^{(1)} \right\}.$$

Starting from  $\mathcal{M}_{1/8}^{(4)}$  configurations and searching for single projectors anticommuting with the ones describing  $\mathcal{M}_{1/8}^{(4)}$  gives rise to these BPS states. The following examples

$$\begin{aligned} &2^3 5^3 \{5, 0, 1, 2, 1, 0\}, \\ &2^3 5^1 6^2 \{3, 1, 3, 1, 1, 0\}, \\ &5^3 6^3 \{1, 1, 4, 2, 2, 0\}, \end{aligned}$$

were generated from (52).

Concerning  $\mathcal{M}_{1/8}^{(a+5)}$ , they are non-threshold bound states depending on  $N_{a+5} = N_a + 1$  integers, built from

$\mathcal{M}_{1/2}$  and  $\mathcal{M}_{1/8}^{(a)}$  subsystems. As such, the prescription to generate these states is the same as the one explained for  $\mathcal{M}_{1/8}^{(4)}$  and  $\mathcal{M}_{1/8}^{(5)}$ . They are described by three projection conditions : two of them characterizing the  $\mathcal{M}_{1/8}^{(a)}$  subsystem and a third one associated with its non-threshold nature.

#### D. $\nu = \frac{1}{16}, \frac{1}{32}$ BPS states

We would like to conclude with a general discussion concerning  $\nu = \frac{1}{16}, \frac{1}{32}$  BPS states extending previous techniques. From the first factoring theorem, whenever  $\nu = \frac{1}{16} = \nu_1 \cdot \nu_2 \Rightarrow \nu_1 = \frac{1}{2}, \nu_2 = \frac{1}{8}$  and  $\nu_1 = \nu_2 = \frac{1}{4}$ , whereas for  $\nu = \frac{1}{32} \Rightarrow \nu_1 = \frac{1}{2}, \nu_2 = \frac{1}{16}$  and  $\nu_1 = \frac{1}{4}, \nu_2 = \frac{1}{32}$ . There will thus be BPS masses of the form

$$\mathcal{M}_{1/16}^{(i)} = \mathcal{M}_{1/2} + \mathcal{M}_{1/8}^{(i)} \quad (54)$$

$$\mathcal{M}_{1/16}^{(ij)} = \mathcal{M}_{1/4}^{(i)} + \mathcal{M}_{1/4}^{(j)} \quad (55)$$

$$\mathcal{M}_{1/32}^{(i)} = \mathcal{M}_{1/2} + \mathcal{M}_{1/16}^{(i)} \quad (56)$$

$$\mathcal{M}_{1/32}^{(ij)} = \mathcal{M}_{1/4}^{(i)} + \mathcal{M}_{1/8}^{(j)}, \quad (57)$$

whose projection conditions are given by the union of the subsystem ones (see (26)).

If we apply the second factoring theorem,  $\nu = 2\nu_1 \cdot \nu_2$ . For  $\nu = \frac{1}{16} \Rightarrow \nu_1 = \frac{1}{4}, \nu_2 = \frac{1}{8}$  or  $\nu_1 = \frac{1}{2}, \nu_2 = \frac{1}{16}$ . On the other hand, when  $\nu = \frac{1}{32} \Rightarrow \nu_1 = \frac{1}{4}, \nu_2 = \frac{1}{16}$ ,  $\nu_1 = \nu_2 = \frac{1}{8}$  or  $\nu_1 = \frac{1}{2}, \nu_2 = \frac{1}{32}$ . We summarize the table of BPS masses corresponding to the above discussion as follows

$$\mathcal{M}_{1/16}^{(ij)} = \sqrt{\left(\mathcal{M}_{1/4}^{(i)}\right)^2 + \left(\mathcal{M}_{1/8}^{(j)}\right)^2} \quad (58)$$

$$\mathcal{M}_{1/16}^{(i)} = \sqrt{\left(\mathcal{M}_{1/2}\right)^2 + \left(\mathcal{M}_{1/16}^{(i)}\right)^2} \quad (59)$$

$$\mathcal{M}_{1/32}^{(ij)} = \sqrt{\left(\mathcal{M}_{1/4}^{(i)}\right)^2 + \left(\mathcal{M}_{1/16}^{(j)}\right)^2} \quad (60)$$

$$\mathcal{M}_{1/32}^{(ij)} = \sqrt{\left(\mathcal{M}_{1/8}^{(i)}\right)^2 + \left(\mathcal{M}_{1/8}^{(j)}\right)^2} \quad (61)$$

$$\mathcal{M}_{1/32}^{(i)} = \sqrt{\left(\mathcal{M}_{1/2}\right)^2 + \left(\mathcal{M}_{1/32}^{(i)}\right)^2}. \quad (62)$$

Their projection conditions can be written by applying the same methodology used for  $\mathcal{M}_{1/8}$  non-threshold BPS states. Just to illustrate this general methodology, let us consider two particular examples. To begin with, we shall analyze  $\mathcal{M}_{1/16} = \sqrt{\left(\mathcal{M}_{1/4}^{(1)}\right)^2 + \left(\mathcal{M}_{1/8}^{(7)}\right)^2}$ . This is a  $\nu = \frac{1}{16}$  non-threshold BPS state. It consists on  $\mathcal{M}_{1/4}^{(1)}$  and  $\mathcal{M}_{1/8}^{(7)}$  subsystems. The first subsystem depends on two subsubsystems  $(\mathcal{P}_1, \mathcal{P}_2)$  satisfying

$$\mathcal{P}_1 \mathcal{P}_2 |\alpha\rangle = |\alpha\rangle, \quad (63)$$



whereas the second subsystem depends on five of them ( $\mathcal{P}_3, \dots, \mathcal{P}_7$ ) such that

$$\begin{aligned}\mathcal{P}_4 \tilde{\mathcal{P}}_5 |\alpha\rangle &= |\alpha\rangle \\ \mathcal{P}_6 \mathcal{P}_7 |\alpha\rangle &= |\alpha\rangle\end{aligned}\quad (64)$$

where  $\tilde{\mathcal{P}}_5 = \frac{1}{\mathcal{M}_{1/4}^{(2)}} \left\{ \mathcal{P}_5 \mathcal{M}_{1/2} + \mathcal{P}_6 \mathcal{M}_{1/4}^{(1)} \right\}$ . The final projection condition describing the non-threshold nature of  $\mathcal{M}_{1/16}$  is given by

$$\mathcal{P}_0 |\alpha\rangle = |\alpha\rangle, \quad (65)$$

where  $\mathcal{P}_0 = \frac{1}{\mathcal{M}_{1/16}^{(2)}} \left\{ \mathcal{P}_1 \mathcal{M}_{1/4}^{(1)} + \mathcal{P}_3 \mathcal{M}_{1/2} + \mathcal{P}_4 \mathcal{M}_{1/8}^{(2)} \right\}$ . One brane configuration satisfying such requirements is given by  $2^2 5^3 6^2 \{2, 1, 3, 3, 0, 1, 0\}$  corresponding to the array

$$\begin{aligned}M5: & 1 \ 2 \ 3 \ 4 \ 5 \ - \ - \ - \ - \ - \\ Mkk: & 1 \ 2 \ 3 \ - \ - \ 6 \ 7 \ 8 \ - \ - \\ M5: & - \ - \ - \ 4 \ 5 \ 6 \ 7 \ - \ 9 \ - \\ M5: & 1 \ - \ - \ 4 \ 5 \ 6 \ - \ 8 \ - \ - \\ M2: & 1 \ 2 \ - \ - \ - \ - \ - \ - \ - \ - \\ M2: & 1 \ - \ 3 \ - \ - \ - \ - \ - \ - \ - \\ Mkk: & 1 \ - \ 3 \ 4 \ 5 \ - \ - \ 8 \ - \ \# \ ,\end{aligned}\quad (66)$$

Finally, consider the  $\nu = \frac{1}{32}$  BPS state with mass  $\mathcal{M}_{1/32} = \sqrt{\left(\mathcal{M}_{1/8}^{(3)}\right)^2 + \left(\mathcal{M}_{1/8}^{(5)}\right)^2}$ . This is a  $\nu = \frac{1}{32}$  non-threshold BPS state. It consists on  $\mathcal{M}_{1/8}^{(3)}$  and  $\mathcal{M}_{1/8}^{(5)}$  subsystems. The first one depends on four subsystems ( $\mathcal{P}_1, \dots, \mathcal{P}_4$ ) satisfying

$$\begin{aligned}\mathcal{P}_1 \mathcal{P}_2 |\alpha\rangle &= |\alpha\rangle \\ \mathcal{P}_3 \mathcal{P}_4 |\alpha\rangle &= |\alpha\rangle\end{aligned}\quad (67)$$

whereas the second subsystem depends on five of them ( $\mathcal{P}_5, \dots, \mathcal{P}_9$ ) such that

$$\begin{aligned}\mathcal{P}_5 \mathcal{P}_6 |\alpha\rangle &= |\alpha\rangle \\ \mathcal{P}_8 \mathcal{P}_9 |\alpha\rangle &= |\alpha\rangle.\end{aligned}\quad (68)$$

The non-threshold nature is given by

$$\begin{aligned}\mathcal{P}_0 |\alpha\rangle &= |\alpha\rangle \\ \mathcal{P}_0 &= \frac{1}{\mathcal{M}_{1/32}} \left\{ \mathcal{P}_1 \mathcal{M}_{1/4}^{(1)} + \mathcal{P}_3 \mathcal{M}_{1/4}'^{(1)} + \mathcal{P}_5 \mathcal{M}_{1/4}'' \right. \\ &\quad \left. + \mathcal{P}_7 \mathcal{M}_{1/2} + \mathcal{P}_8 \mathcal{M}_{1/4}'''^{(1)} \right\}.\end{aligned}\quad (69)$$

An example for this kind of non-threshold bound state is provided by  $2^6 5^2 9^1 \{5, 1, 0, 0, 0, 4, 0, 0, 0\}$  or in terms of its array

$$\begin{aligned}M2: & 1 \ 2 \ - \ - \ - \ - \ - \ - \ - \ - \\ M2: & - \ - \ 3 \ 4 \ - \ - \ - \ - \ - \ - \\ M2: & 1 \ - \ 3 \ - \ - \ - \ - \ - \ - \ - \\ M2: & - \ 2 \ - \ 4 \ - \ - \ - \ - \ - \ - \\ M2: & 1 \ - \ - \ 4 \ - \ - \ - \ - \ - \ - \\ M2: & - \ 2 \ 3 \ - \ - \ - \ - \ - \ - \ - \\ M5: & 1 \ 2 \ 3 \ 4 \ 5 \ - \ - \ - \ - \ - \\ M9: & 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ - \\ M5: & 1 \ 2 \ 3 \ 4 \ - \ - \ - \ - \ - \ \# \ .\end{aligned}\quad (70)$$

#### IV. FURTHER BPS STATES

It should be clear that the set of BPS states described in the previous section is just a subset of the full set of BPS states in M-theory. First of all, the existence of BPS states corresponding to branes intersecting at angles is already known. There has been a lot of work in this direction [14], but the main idea behind their classification was based on the resolution of eq. (3), since one starts from a set of parallel single branes, thus breaking  $\nu = \frac{1}{2}$ , and rotate the projection condition  $R\mathcal{P}_i R^{-1} |\alpha\rangle = |\alpha\rangle$ . In this way, one retains the interpretation as an intersection of branes where the second brane intersects the first one at angles. We will have nothing more to say about these configurations. Besides that, we would like to comment on the possible existence<sup>5</sup> of exotic branes (appearing by fine tuning the value of the central charges) and non-factorizable BPS states, that is, those not classified by the criteria introduced in section III.

##### A. Exotic Branes

In section II, we assumed the values of the charge operators  $\{\mathcal{Z}\}$  were arbitrary, but fixed. It was already pointed out in [7], that for certain configurations, there exist very precise values of  $\{\mathcal{Z}\}$  giving rise to BPS configurations preserving *exotic* fractions of supersymmetry. In particular, they considered the M-theory configuration

$$\begin{aligned}M5: & 1 \ 2 \ 3 \ 4 \ 5 \ - \ - \ - \ - \ - \\ M5: & - \ - \ - \ - \ 5 \ 6 \ 7 \ 8 \ 9 \ - \\ M2: & - \ - \ - \ - \ 5 \ - \ - \ - \ - \ \# \ .\end{aligned}$$

The latter is described by three mutually commuting single projectors ( $\Gamma_{(1)} = \Gamma_{012345}, \Gamma_{(2)} = \Gamma_{056789}, \Gamma_{(3)} = \Gamma_{05\#}$ ), where the third condition  $\Gamma_{(3)} |\alpha\rangle = |\alpha\rangle$  can be derived from  $\Gamma_{(1)} |\alpha\rangle = \Gamma_{(2)} |\alpha\rangle = |\alpha\rangle$ . Equivalently, the M2-brane can be added for free.

<sup>5</sup>By possible, we mean not forbidden by pure algebraic considerations.

Actually, any configuration consisting on three commuting single branes such that one of them is for free will admit exotic fractions of supersymmetry. The proof follows from the analysis done in [7]. We can simultaneously diagonalize the three single projectors. In particular,

$$\begin{aligned}\Gamma_1 &= \text{diag}(1, 1, -1, -1) \otimes \mathbb{I}_8 \\ \Gamma_2 &= \text{diag}(1, -1, 1, -1) \otimes \mathbb{I}_8 \\ \Gamma_3 &= \text{diag}(1, -1, -1, 1) \otimes \mathbb{I}_8,\end{aligned}\quad (71)$$

where we restricted ourselves to  $\Gamma_3 = \Gamma_1\Gamma_2$ , without loss of generality. In this case, the eigenvalue problem is equivalent to

$$\{Q, Q\} = \text{diag}(\mathcal{M} - \lambda_1, \mathcal{M} - \lambda_2, \mathcal{M} - \lambda_3, \mathcal{M} - \lambda_4) \otimes \mathbb{I}_8.$$

The positivity of its left hand side implies that  $\mathcal{M} \geq \lambda_i \forall i$ , where

$$\begin{aligned}\lambda_1 &= \mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Z}_3 \\ \lambda_2 &= -\lambda_1 + 2\mathcal{Z}_1 \\ \lambda_3 &= -\lambda_1 + 2\mathcal{Z}_2 \\ \lambda_4 &= -\lambda_1 + 2\mathcal{Z}_3.\end{aligned}\quad (72)$$

It is clear that  $\lambda_2$  is the biggest eigenvalue whenever  $0 \geq \mathcal{Z}_1 \geq \mathcal{Z}_2$  and  $0 \geq \mathcal{Z}_1 \geq \mathcal{Z}_3$ , so that the BPS bound becomes

$$\mathcal{M} \geq -\lambda_1 + 2\mathcal{Z}_1. \quad (73)$$

When the latter is saturated, and for arbitrary values of  $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$  satisfying the latter restrictions, then

$$\{Q, Q\} = \text{diag}(-2(\mathcal{Z}_2 + \mathcal{Z}_3), 0, 2(\mathcal{Z}_1 - \mathcal{Z}_2), 2(\mathcal{Z}_1 - \mathcal{Z}_3)) \otimes \mathbb{I}_8 \quad (74)$$

the corresponding BPS state preserves 1/4 of susy. Notice that whenever  $\mathcal{Z}_1 = \mathcal{Z}_2$  or  $\mathcal{Z}_1 = \mathcal{Z}_3$ , the amount of susy becomes enhanced to 1/2, and the mass becomes  $\mathcal{M} = |\mathcal{Z}_3|$  or  $\mathcal{M} = |\mathcal{Z}_2|$ , respectively. Furthermore, when  $\mathcal{Z}_1 = \mathcal{Z}_2 = \mathcal{Z}_3$ , the amount of susy preserved is 3/4 and the mass is given by  $\mathcal{M} = |\mathcal{Z}_1|$ .

Thus, any intersection of branes described in table II can give rise to a exotic brane configuration just by considering as the third “free” brane the one obtained from the product of the initial single projectors. Among this set of configurations one can find

$$\begin{aligned}M5: & \quad 1 \ 2 \ 3 \ 4 \ 5 \ - \ - \ - \ - \\ Mkk: & \quad - \ - \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ - \ - \\ M5: & \quad - \ - \ 3 \ 4 \ 5 \ - \ - \ - \ 9 \ \# \ ,\end{aligned}$$

or

$$\begin{aligned}Mkk: & \quad 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ - \ - \ - \ - \\ Mkk: & \quad - \ - \ - \ - \ 5 \ 6 \ 7 \ 8 \ 9 \ \# \\ M2: & \quad - \ - \ - \ - \ 5 \ 6 \ - \ - \ - \ - \ .\end{aligned}$$

There will generically be an enhancement of supersymmetry by fine tuning the charges  $\{\mathcal{Z}\}$  whenever some of the commuting single projectors involved in the configuration are dependent. Such enhancements may include new exotic fractions of supersymmetry. To illustrate this point, we will prove that such an enhancement is not possible whenever we have a set of  $n$  independent commuting single projectors ( $n \leq 5$ ).

As usual, we can simultaneously diagonalize the set of single projectors giving rise to a set of  $2^n$  eigenvalues

$$\lambda_i = \pm \mathcal{Z}_1 \pm \dots \pm \mathcal{Z}_n.$$

Let  $\tilde{\lambda} \geq \lambda_i$ , where  $\tilde{\lambda} = \tilde{\epsilon}_1 \mathcal{Z}_1 + \dots + \tilde{\epsilon}_n \mathcal{Z}_n$ . When the BPS bound is saturated,  $\mathcal{M} = \tilde{\lambda}$  and

$$\tilde{\lambda} - \lambda_i = (\tilde{\epsilon}_1 - \epsilon_i) \mathcal{Z}_1 + \dots + (\tilde{\epsilon}_n - \epsilon_i) \mathcal{Z}_n \geq 0.$$

By hypothesis,  $\mathcal{Z}_i = \tilde{\epsilon}_i |\mathcal{Z}_i|$ , so that

$$\tilde{\lambda} - \lambda_i = \sum_{i=1}^n (1 - \tilde{\epsilon}_i \epsilon_i) |\mathcal{Z}_i| \geq 0.$$

Whenever signs coincide ( $\tilde{\epsilon}_i = \epsilon_i$ ), there is no contribution to the left hand side summation. Whenever they do not coincide,  $\tilde{\epsilon}_i \epsilon_i = -1$  so we are left with a sum of positively defined terms ( $2|\mathcal{Z}_i|$ ). Thus the only possibility to satisfy the equality in the previous equation is to switch off the charges  $|\mathcal{Z}_i| = 0$ , which corresponds to not having these branes in the configuration. We conclude that it is not possible to enhance supersymmetry by fine tuning of charges whenever the set of  $n$  commuting single projectors are independent, as we claimed. It is straightforward to show that the latter conclusion does not apply when some of the single projectors are dependent, as explicitly proved at the beginning of this subsection in a particular case.

## B. Non-factorizable states

In this subsection, we shall provide an example for a non-factorizable BPS state. As it has already been stressed in section II, one could have studied the classification of BPS states in terms of the number of single branes building up the state ( $N$ ) and the number of commutation relations among the single projectors involved in (3). The easiest non-factorizable system that one faces is found for  $N = 4$  and it is described by the commutation relations :

$$\begin{aligned}[\Gamma_1, \Gamma_2] &= [\Gamma_1, \Gamma_3] = [\Gamma_3, \Gamma_4] = 0, \\ \{\Gamma_1, \Gamma_4\} &= \{\Gamma_2, \Gamma_3\} = \{\Gamma_2, \Gamma_4\} = 0\end{aligned}\quad (75)$$

One may proceed as in previous subsections to solve the eigenvalue problem (3), and one would derive that such state  $|\alpha\rangle$  exists and satisfies

$$\mathcal{P}_1|\alpha\rangle = \frac{\mathcal{Z}_1\Gamma_1 + \mathcal{Z}_2\Gamma_2 + \mathcal{Z}_3\Gamma_3 + \mathcal{Z}_4\Gamma_4}{\mathcal{M}}|\alpha\rangle \quad (76)$$

$$\mathcal{P}_2|\alpha\rangle = \frac{2\mathcal{Z}_1\mathcal{Z}_2\Gamma_1\Gamma_2 + 2\mathcal{Z}_1\mathcal{Z}_3\Gamma_1\Gamma_3 + 2\mathcal{Z}_3\mathcal{Z}_4\Gamma_3\Gamma_4}{\mathcal{M}'}|\alpha\rangle \quad (77)$$

whereas its mass is given by

$$\mathcal{M} = \sqrt{\mathcal{Z}_1^2 + \mathcal{Z}_2^2 + \mathcal{Z}_3^2 + \mathcal{Z}_4^2 + \mathcal{M}'} \quad (78)$$

$$\mathcal{M}' = \sqrt{(2\mathcal{Z}_1\mathcal{Z}_2)^2 + (2\mathcal{Z}_1\mathcal{Z}_3)^2 + (2\mathcal{Z}_3\mathcal{Z}_4)^2}. \quad (79)$$

At this stage, one is left to determine the amount of supersymmetry preserved by  $|\alpha\rangle$ . In this case, the analysis is not so straightforward as in the factorizable case because even though  $\mathcal{P}_2^2 = \mathbb{I}$  and  $\text{tr}\mathcal{P}_2 = 0$ , the same does not hold for  $\mathcal{P}_1$ . Its trace is still vanishing, but

$$\mathcal{P}_1^2 = a\mathbb{I} + b\mathcal{P}_2, \quad (80)$$

where  $a = \sum_{i=1}^4 \mathcal{Z}_i^2 / \mathcal{M}^2$  and  $b = \mathcal{M}' / \mathcal{M}^2$ , so that  $a+b = 1$ . It is nevertheless still true that  $[\mathcal{P}_1, \mathcal{P}_2] = 0$ , so that they can still be simultaneously diagonalized. Take

$$\mathcal{P}_2 = \text{diag}(\underbrace{1, \dots, 1}_{16}, \underbrace{-1, \dots, -1}_{16}).$$

The latter fixes  $\mathcal{P}_1^2$  through (80), and since  $\mathcal{P}_1$  is also diagonal in this basis,  $\mathcal{P}_1$  is determined modulo signs. Actually,  $\mathcal{P}_1$  depends on the number of plus signs appearing in the first and second  $16 \times 16$  subspaces ( $N_1, N_2$ ) and also in  $a$  or  $b$  parameters. The constraint provided by the vanishing of the trace gives

$$N_1 + N_2\sqrt{1-2b} = 8(1 + \sqrt{1-2b}). \quad (81)$$

Whenever  $0 \leq b \leq \frac{1}{2}$ , but otherwise arbitrary, the solution to equation (81) involves  $N_1 = 8$ <sup>6</sup>. Altogether, we conclude that there exist  $\nu = \frac{1}{4}$  non-decouppable BPS states, which were certainly not included in our previous classification. A very simple configuration satisfying the commutation relations (75) is

$$\begin{array}{llllllll} M2 : & 1 & 2 & - & - & - & - & - \\ M2 : & - & 2 & 3 & - & - & - & - \\ M2 : & - & - & 3 & 4 & - & - & - \\ M2 : & - & - & - & 4 & 5 & - & - \end{array}$$

<sup>6</sup>The purpose of this subsection is to illustrate the existence of solutions to the eigenvalue problem (3) corresponding to non-decouppable states, and not to give a full analysis to the projection conditions (76) and (77).

## V. AUTOMORPHISMS

It has already been pointed out [8,9,12] that the maximal automorphism group of the  $\mathcal{N} = 1$   $D = 11$  Super-Poincaré algebra is  $GL(32, \mathbb{R})$ . Let us write the forementioned algebra as

$$\{Q_\alpha, Q_\beta\} = \mathcal{Z}_{\alpha\beta} \quad , \quad [Q_\gamma, \mathcal{Z}_{\alpha\beta}] = [\mathcal{Z}_{\alpha\beta}, \mathcal{Z}_{\gamma\delta}] = 0.$$

If we consider the most general transformation on the supercharges,  $Q'_\alpha = (U Q)_\alpha$ ,  $U \in GL(32, \mathbb{R})$ , the latter will indeed be an automorphism of the algebra if

$$\mathcal{Z}'_{\alpha\beta} = (U \mathcal{Z} U^t)_{\alpha\beta}. \quad (82)$$

As stressed in [8], any generator of the  $GL(32, \mathbb{R})$  transformation can be expanded in terms of the elements of the enveloping Clifford algebra. This gives rise to  $2^{10}$  independent generators, matching the dimension of  $GL(32, \mathbb{R})$ . In particular, one may consider elements of the subgroup  $SL(32, \mathbb{R})$  of the form  $U = e^{\alpha\Gamma/2}$  where  $\Gamma = \Gamma_{i_1 \dots i_p}$ ,  $i_j = 1, \dots, 9, \#$ ,  $j \leq 1, \dots, p$  and  $p \leq \#$ . They are indeed elements of  $SL(32, \mathbb{R})$  since  $\det U = 1$ <sup>7</sup>. On the other hand, there are  $2^{10} - 1$  independent elements of this type since

$$\sum_{i=1}^{10} \binom{10}{i} = 2^{10} - 1,$$

the missing element being the identity. It will be important for the rest of the discussion to distinguish among those  $\Gamma$ 's being symmetric  $\Gamma_s$  ( $p = 1, 4, 5, 8, 9$ ) and those being antisymmetric  $\Gamma_a$  ( $p = 2, 3, 6, 7, 10$ ). It is clear that  $\Gamma_s^2 = 1$ , whereas  $\Gamma_a^2 = -1$ , so that we can write

$$U_s = e^{\alpha\Gamma_s/2} = \cosh \frac{\alpha}{2} + \sinh \frac{\alpha}{2} \Gamma_s \quad (83)$$

$$U_a = e^{\alpha\Gamma_a/2} = \cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} \Gamma_a. \quad (84)$$

Furthermore, the subset of elements  $U_a$  belongs to an  $SO(32)$  subgroup which leaves the mass operator invariant. That they belong to  $SO(32)$  is clear because  $U_a^t = U_a^{-1}$  and that they leave the mass operator invariant is also clear since

$$U_a \Gamma U_a^t = \Gamma \quad (85)$$

whenever  $[\Gamma, \Gamma_a] = 0$ , which is the case for the mass operator  $-\mathcal{M}\delta_{\alpha\beta}$ .

<sup>7</sup>Notice that we only allowed spacelike indexes in the antisymmetrized gamma matrices defining the group element  $U$ , so that  $\text{tr}\Gamma = 0$ , which guarantees  $\det U = e^{\text{tr}\log U} = e^{\alpha \text{tr}\Gamma/2} = 1$ .

Using this formulation is particularly useful to show that  $U$  transformations leave indeed the *superalgebra covariant*. Notice that  $\mathcal{Z}_{\alpha\beta}$  is expanded in terms of symmetric matrices belonging to the enveloping Clifford algebra  $(\Gamma_s)$ . We already know that these matrices are left invariant under  $U_a$  if  $[\Gamma_s, \Gamma_a] = 0$ . When  $\{\Gamma_s, \Gamma_a\} = 0$ , then

$$U_a \Gamma_s U_a^t = \cos \alpha \Gamma_s + \sin \alpha \Gamma_a \Gamma_s. \quad (86)$$

Due to the fact that  $(\Gamma_a \Gamma_s)^2 = \mathbb{I}$  and  $(\Gamma_a \Gamma_s)^t = \Gamma_a \Gamma_s$ , one can always write  $\Gamma_a \Gamma_s$  as a symmetric matrix  $\Gamma_a \Gamma_s = \Gamma'_s$ .

On the other hand,  $\Gamma_s$  is left invariant under  $U_s = e^{\alpha \Gamma'_s/2}$  whenever  $\{\Gamma_s, \Gamma'_s\} = 0$ . If  $[\Gamma_s, \Gamma'_s] = 0$ , then

$$U_s \Gamma_s U_s^t = \cosh \alpha \Gamma_s + \sinh \alpha \Gamma_s \Gamma'_s. \quad (87)$$

Again  $\Gamma_s \Gamma'_s = \Gamma''_s$  is a symmetric matrix. Notice, in particular, that the mass operator transforms under  $U_s$  transformations.

Thus  $U \mathcal{Z} U^t$  is a combination of symmetric matrices which upon requiring the matching with  $\mathcal{Z}'_{\alpha\beta}$  will induce some transformations on the “central charges”  $\mathcal{Z}_{M\dots}$ . For example, let us consider  $U_a \in SO(32)$  and its effect on  $\mathcal{Z}_s \Gamma_s + \mathcal{Z}'_s \Gamma'_s$ , where  $\Gamma'_s = \Gamma_s \Gamma_a$  and  $\{\Gamma_s, \Gamma_a\} = 0$ . An straightforward computation shows that

$$U_a (\mathcal{Z}_s \Gamma_s + \mathcal{Z}'_s \Gamma'_s) U_a^t = (\mathcal{Z}_s \cos \alpha + \mathcal{Z}'_s \sin \alpha) \Gamma_s + (-\mathcal{Z}_s \sin \alpha + \mathcal{Z}'_s \cos \alpha) \Gamma'_s. \quad (88)$$

The latter induces an  $SO(2)$  transformation on the space of charges expanded by  $\mathcal{Z}_s$  and  $\mathcal{Z}'_s$ . Indeed,

$$\begin{pmatrix} \tilde{\mathcal{Z}}_s \\ \tilde{\mathcal{Z}}'_s \end{pmatrix} = R \begin{pmatrix} \mathcal{Z}_s \\ \mathcal{Z}'_s \end{pmatrix} \quad (89)$$

where

$$R = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \in SO(2). \quad (90)$$

Analogously, we could have considered  $U_s = e^{\alpha \Gamma'_s/2}$  and its effect on  $\mathcal{Z}_s \Gamma_s + \mathcal{Z}'_s \Gamma'_s$ , where  $\Gamma'_s = \Gamma_s \Gamma''_s$  and  $[\Gamma_s, \Gamma''_s] = 0$ . As before,

$$U_s (\mathcal{Z}_s \Gamma_s + \mathcal{Z}'_s \Gamma'_s) U_s^t = (\mathcal{Z}_s \cosh \alpha + \mathcal{Z}'_s \sinh \alpha) \Gamma_s + (\mathcal{Z}_s \sinh \alpha + \mathcal{Z}'_s \cosh \alpha) \Gamma'_s, \quad (91)$$

which in this case involves an  $SO(1,1)$  transformation on the space of charges expanded by  $\mathcal{Z}_s$  and  $\mathcal{Z}'_s$ . Indeed,

$$\begin{pmatrix} \tilde{\mathcal{Z}}_s \\ \tilde{\mathcal{Z}}'_s \end{pmatrix} = S \begin{pmatrix} \mathcal{Z}_s \\ \mathcal{Z}'_s \end{pmatrix} \quad (92)$$

where

$$S = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix} \in SO(1,1). \quad (93)$$

Let us reexamine eq. (86) when  $\Gamma_a = \Gamma'_s \Gamma_s$ . Since  $\{\Gamma_a, \Gamma_s\} = 0$  by hypothesis,  $\{\Gamma_s, \Gamma'_s\} = 0$  and eq. (86) looks like

$$U_a \Gamma_s U_a^t = \cos \alpha \Gamma_s + \sin \alpha \Gamma'_s, \quad (94)$$

this being nothing else but a linear combination of two anticommuting single projectors  $(\Gamma_s, \Gamma'_s)$  with coefficients parametrizing a circle. The latter was the operator  $\mathcal{P}$  describing a non-threshold bound state  $|\alpha\rangle$ . Since  $|\alpha\rangle$  is a Clifford valued state, we can look at  $\mathcal{P}|\alpha\rangle = |\alpha\rangle$  as

$$U_a \Gamma_s U_a^t |\alpha\rangle = |\alpha\rangle \Leftrightarrow \Gamma_s (U_a^t |\alpha\rangle) = (U_a^t |\alpha\rangle), \quad (95)$$

which tells us that the initial  $\nu = \frac{1}{2}$  non-threshold bound state  $|\alpha\rangle$  is  $SO(32)$  related with a  $\nu = \frac{1}{2}$  single brane (at threshold)  $|\alpha'\rangle = U_a^t |\alpha\rangle$  having the same mass.

As a first consequence, we can immediately state that any non-threshold bound state appearing in table III is  $SO(32)$  related with a  $\nu = \frac{1}{2}$  bound state at threshold. Let us analyze some particular examples of this phenomena. Consider the configuration

$$\begin{array}{l} M2: 1 \ 2 \ - \ - \ - \ - \ - \ - \ - \\ M2: - \ 2 \ 3 \ - \ - \ - \ - \ - \ - \end{array}$$

characterized by  $\mathcal{P} = \cos \alpha \Gamma_{012} + \sin \alpha \Gamma_{023}$  and  $\mathcal{M} = \sqrt{\mathcal{Z}_{12}^2 + \mathcal{Z}_{23}^2}$ . It is  $SO(32)$  related with a single M2-brane of the same mass through  $U_\alpha = e^{\alpha \Gamma_{13}/2}$ , which corresponds to a rotation of angle  $\alpha = \arccos(\frac{\mathcal{Z}_{12}}{\mathcal{M}})$  in the 13-plane.

There exist much more exotic transformations such as the one relating

$$\begin{array}{l} M5: 1 \ 2 \ 3 \ 4 \ 5 \ - \ - \ - \ - \ - \\ M2: 1 \ 2 \ - \ - \ - \ - \ - \ - \ - \end{array}$$

to a single M5-brane through  $U_\alpha = e^{\alpha \Gamma_{345}/2}$ , which is reminiscent of the electro-magnetic duality transformation in eight dimensions (dyonic membranes) [15] or the one relating

$$\begin{array}{l} Mkk: 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ - \ - \ - \ - \\ M2: - \ - \ - \ - \ - \ 6 \ 7 \ - \ - \ - \end{array}$$

to a single Mkk-monopole through  $U_\alpha = e^{\alpha \Gamma_{123457}/2}$ .

Actually, this construction can be extended to a linear combination of an arbitrary number  $(n+1)$  of mutually anticommuting single projectors, with coefficients parametrizing  $S_n$ . Indeed, consider

$$\mathcal{P} = \cos \alpha_1 \Gamma_1 + \sin \alpha_1 (\cos \alpha_2 \Gamma_2 + \sin \alpha_2 (\dots + \sin \alpha_{n-1} (\cos \alpha_n \Gamma_n + \sin \alpha_n \Gamma_{n+1}) \dots)) .$$

Using identity (94) iteratively,

$$\begin{aligned}\cos \alpha_n \Gamma_n + \sin \alpha_n \Gamma_{n+1} &= U_{\alpha_n} \Gamma_n, \\ U_{\alpha_n} &= e^{\alpha_n \Gamma_{n+1} \Gamma_n}\end{aligned}\quad (96)$$

$$\begin{aligned}\cos \alpha_{n-1} \Gamma_{n-1} + \sin \alpha_{n-1} U_{\alpha_n} \Gamma_n &= U_{\alpha_{n-1}} \Gamma_{n-1}, \\ U_{\alpha_{n-1}} &= e^{\alpha_{n-1} U_{\alpha_n} \Gamma_n \Gamma_{n-1}}\end{aligned}\quad (97)$$

$$\begin{aligned}\cos \alpha_1 \Gamma_1 + \sin \alpha_1 U_{\alpha_2} \Gamma_2 &= U_{\alpha_1} \Gamma_1, \\ U_{\alpha_1} &= e^{\alpha_1 U_{\alpha_2} \Gamma_2 \Gamma_1},\end{aligned}\quad (98)$$

we can rewrite  $\mathcal{P}$  as

$$\mathcal{P} = U_{\alpha_1} \Gamma_1 = U_{\alpha_1/2} \Gamma_1 U_{\alpha_1/2}^t. \quad (99)$$

Thus any non-threshold bound state preserving  $\nu = \frac{1}{2}$  characterized by  $\mathcal{P}|\alpha\rangle = |\alpha\rangle$  is  $SO(32)$  related with a  $\nu = \frac{1}{2}$  bound state at threshold having the same mass.

We shall now move to less supersymmetric BPS states and study whether we can find more involved  $SO(32)$  transformations relating non-threshold bound states to bound states at threshold. Let us start by BPS states described by (33) and (35). Without loss of generality, we shall concentrate on operators  $\mathcal{P}_i$  collapsing to single projectors ( $\mathcal{P}_i = \Gamma_i$ )

$$\begin{aligned}\Gamma_3 \Gamma_4 |\alpha\rangle &= |\alpha\rangle \\ (\cos \hat{\alpha} \Gamma_1 + \sin \hat{\alpha} \Gamma_3) |\alpha\rangle &= |\alpha\rangle,\end{aligned}\quad (100)$$

where  $\tan \hat{\alpha} = \frac{\mathcal{Z}_3 + \mathcal{Z}_4}{\mathcal{Z}_1}$  and whose mass is given by

$$\mathcal{M} = \sqrt{(\mathcal{Z}_1)^2 + (\mathcal{Z}_3 + \mathcal{Z}_4)^2}. \quad (101)$$

Let us compute the transformed charges ( $\mathcal{Z}'$ ) under the finite  $SO(32)$  transformation

$$U = e^{\alpha \Gamma_3 \Gamma_1/2} e^{\beta \Gamma_4 \Gamma_1/2}$$

for arbitrary  $\alpha, \beta$  parameters. Using (82), we obtain

$$\begin{aligned}\mathcal{Z}'_1 &= \mathcal{Z}_1 \cos \alpha \cos \beta - \mathcal{Z}_3 \sin \alpha \cos \beta - \mathcal{Z}_4 \cos \alpha \sin \beta \\ \mathcal{Z}'_3 &= \mathcal{Z}_1 \sin \alpha \cos \beta + \mathcal{Z}_3 \cos \alpha \cos \beta - \mathcal{Z}_4 \sin \alpha \sin \beta \\ \mathcal{Z}'_4 &= \mathcal{Z}_1 \cos \alpha \sin \beta - \mathcal{Z}_3 \sin \alpha \sin \beta + \mathcal{Z}_4 \cos \alpha \cos \beta \\ \mathcal{Z}'_{134} &= -\mathcal{Z}_1 \sin \alpha \sin \beta - \mathcal{Z}_3 \cos \alpha \sin \beta - \mathcal{Z}_4 \sin \alpha \cos \beta,\end{aligned}\quad (102)$$

where  $\mathcal{Z}'_{134}$  is the central charge associated with the single projector  $\Gamma_1 \Gamma_3 \Gamma_4$ . We are thus led to four independent charges. Actually, we can appropriately fix  $\alpha$  and  $\beta$  to set two of the  $\mathcal{Z}'$ 's to zero. In particular, from the requirements  $\mathcal{Z}'_3 + \mathcal{Z}'_4 = 0$  and  $\mathcal{Z}'_3 - \mathcal{Z}'_4 = 0$ , we derive two conditions

$$\tan(\alpha + \beta) = -\frac{\mathcal{Z}_3 + \mathcal{Z}_4}{\mathcal{Z}_1} \quad (103)$$

$$\tan(\alpha - \beta) = \frac{\mathcal{Z}_4 - \mathcal{Z}_3}{\mathcal{Z}_1}, \quad (104)$$

fixing both arbitrary parameters. Notice that  $\hat{\alpha} = -(\alpha + \beta)$ . It can be checked that the invariant mass  $\mathcal{M}$ , when expressed in terms of the transformed charges, looks as

$$\mathcal{M} = \mathcal{Z}'_1 + \mathcal{Z}'_{134},$$

which is reminiscent of a  $\nu = \frac{1}{4}$  BPS state at threshold. This expectation can be further confirmed by analysing the projection conditions satisfied by  $|\alpha'\rangle = U^t |\alpha\rangle$ . Since  $[\Gamma_3 \Gamma_4, U] = 0$ ,

$$\Gamma_3 \Gamma_4 |\alpha'\rangle = |\alpha'\rangle. \quad (105)$$

On the other hand,

$$\begin{aligned}U^t (\cos \hat{\alpha} \Gamma_1 + \sin \hat{\alpha} \Gamma_3) U &= \\ \cos(\hat{\alpha} + \alpha + \beta) \Gamma_1 + \sin(\hat{\alpha} + \alpha + \beta) \Gamma_3 &= \Gamma_1\end{aligned}\quad (106)$$

thus,

$$\begin{aligned}U^t (\cos \hat{\alpha} \Gamma_1 + \sin \hat{\alpha} \Gamma_3) U |\alpha'\rangle &= |\alpha'\rangle \\ \Downarrow \\ \Gamma_1 |\alpha'\rangle &= |\alpha'\rangle.\end{aligned}\quad (107)$$

Joining equations (105) and (107), the transformed BPS state is described by

$$\begin{aligned}\Gamma_1 |\alpha'\rangle &= |\alpha'\rangle \\ \Gamma_1 \Gamma_3 \Gamma_4 |\alpha'\rangle &= |\alpha'\rangle \\ \mathcal{M} &= \mathcal{Z}'_1 + \mathcal{Z}'_{134},\end{aligned}\quad (108)$$

corresponding to a  $\nu = \frac{1}{4}$  BPS state at threshold.

Even though the latter proof applies to any set of  $\{\Gamma_1, \Gamma_3, \Gamma_4\}$  satisfying  $\{\Gamma_1, \Gamma_3\} = \{\Gamma_1, \Gamma_4\} = [\Gamma_3, \Gamma_4] = 0$ , it may be useful to consider some particular examples. One involving just  $SO(10)$  rotations is

$$\begin{array}{cccccccc}M2: & - & 2 & 3 & - & - & - & - \\M2: & 1 & 2 & - & - & - & - & - \\M2: & - & - & 3 & 4 & - & - & -\end{array}$$

This non-threshold  $\nu = \frac{1}{4}$  BPS state is related by the transformation

$$\begin{aligned}U &= e^{-\alpha \Gamma_{13}/2} e^{\beta \Gamma_{24}/2} \\ \tan(\alpha + \beta) &= -\frac{\mathcal{Z}_{12} + \mathcal{Z}_{34}}{\mathcal{Z}_{23}} \\ \tan(\alpha - \beta) &= \frac{\mathcal{Z}_{34} - \mathcal{Z}_{12}}{\mathcal{Z}_{23}},\end{aligned}\quad (109)$$

to the BPS state at threshold described by

$$\begin{aligned}\Gamma_{023} |\alpha'\rangle &= |\alpha'\rangle \\ \Gamma_{014} |\alpha'\rangle &= |\alpha'\rangle \\ \mathcal{M} &= \mathcal{Z}'_{23} + \mathcal{Z}'_{14}\end{aligned}\quad (110)$$

corresponding to the array  $M2 \perp M2(0)$

$$\begin{array}{l} M2 : \_ \ 2 \ 3 \ \_ \ \_ \ \_ \ \_ \ \_ \ \_ \\ M2 : 1 \ \_ \ \_ \ 4 \ \_ \ \_ \ \_ \ \_ \ \_ \end{array}$$

Another less trivial example is provided by the array

$$\begin{array}{l} M2 : 1 \ 2 \ \_ \ \_ \ \_ \ \_ \ \_ \ \_ \\ M5 : 1 \ \_ \ 3 \ 4 \ 5 \ 6 \ \_ \ \_ \ \_ \\ M2 : 1 \ \_ \ 3 \ \_ \ \_ \ \_ \ \_ \ \_ \end{array}$$

The latter can be SO(32) related to

$$\begin{array}{l} M2 : 1 \ \_ \ 3 \ \_ \ \_ \ \_ \ \_ \ \_ \\ M5 : 1 \ 2 \ \_ \ 4 \ 5 \ 6 \ \_ \ \_ \ \_ \end{array},$$

through  $U = e^{\alpha\Gamma_{23}/2}e^{-\beta\Gamma_{456}/2}$  with  $\alpha, \beta$  satisfying

$$\begin{aligned} \tan(\alpha + \beta) &= -\frac{\mathcal{Z}_{12} + \mathcal{Z}_{13456}}{\mathcal{Z}_{13}} \\ \tan(\alpha - \beta) &= \frac{\mathcal{Z}_{13456} - \mathcal{Z}_{12}}{\mathcal{Z}_{13}}. \end{aligned} \quad (111)$$

The above construction can be extended to non-threshold  $\nu = \frac{1}{8}$  BPS states described by

$$\begin{aligned} \Gamma_1\Gamma_2|\alpha\rangle &= |\alpha\rangle \\ \Gamma_3\Gamma_4|\alpha\rangle &= |\alpha\rangle \\ (\cos\hat{\alpha}\Gamma_1 + \sin\hat{\alpha}\Gamma_3)|\alpha\rangle &= |\alpha\rangle \end{aligned} \quad (112)$$

whose mass is given by

$$\mathcal{M} = \sqrt{(\mathcal{Z}_1 + \mathcal{Z}_2)^2 + (\mathcal{Z}_3 + \mathcal{Z}_4)^2}. \quad (113)$$

Proceeding as in the previous case, one can compute the transformed charges ( $\mathcal{Z}'$ ) under the finite SO(32) transformation

$$U = e^{\alpha\Gamma_3\Gamma_1/2}e^{\beta\Gamma_4\Gamma_1/2}e^{\gamma\Gamma_3\Gamma_2/2}e^{\delta\Gamma_4\Gamma_2/2}$$

for arbitrary  $\alpha, \beta, \gamma, \delta$  parameters. The number of independent transformed charges ( $\mathcal{Z}'$ ) is eight. The new four ones  $\{\mathcal{Z}'_{123}, \mathcal{Z}'_{124}, \mathcal{Z}'_{134}, \mathcal{Z}'_{234}\}$  are associated with the single projectors  $\{\Gamma_1\Gamma_2\Gamma_3, \Gamma_1\Gamma_2\Gamma_4, \Gamma_1\Gamma_3\Gamma_4, \Gamma_2\Gamma_3\Gamma_4\}$  respectively. Requiring

$$\mathcal{Z}'_3 + \epsilon_1\mathcal{Z}'_4 + \mathcal{Z}'_{123} + \epsilon_2\mathcal{Z}'_{124} = 0$$

$\forall \epsilon_1, \epsilon_2$  satisfying  $\epsilon_1^2 = \epsilon_2^2 = 1$ , one can fix the four parameters

$$\tan[\alpha + \epsilon_1\beta + \epsilon_2(\gamma + \epsilon_1\delta)] = -\frac{\mathcal{Z}_3 + \epsilon_1\mathcal{Z}_4}{\mathcal{Z}_1 + \epsilon_2\mathcal{Z}_2}.$$

At this stage, one is left with four non-vanishing charges  $\{\mathcal{Z}'_1, \mathcal{Z}'_2, \mathcal{Z}'_{134}, \mathcal{Z}'_{234}\}$  associated with the four commuting single projectors  $\{\Gamma_1, \Gamma_2, \Gamma_1\Gamma_3\Gamma_4, \Gamma_2\Gamma_3\Gamma_4\}$ . When reexpressing the mass in terms of these charges, one derives

$$\mathcal{M} = \mathcal{Z}'_1 + \mathcal{Z}'_2 + \mathcal{Z}'_{134} + \mathcal{Z}'_{234}.$$

All the above information seems to indicate that the transformed state preserves  $\nu = \frac{1}{16}$ , but this conclusion is incorrect. If one analyses the transformed supersymmetry projection conditions satisfied by  $|\alpha'\rangle = U^t|\alpha\rangle$  one is left with

$$\begin{aligned} \Gamma_1|\alpha'\rangle &= |\alpha'\rangle & \Gamma_1|\alpha'\rangle &= |\alpha'\rangle \\ \Gamma_1\Gamma_2|\alpha'\rangle &= |\alpha'\rangle & \Leftrightarrow \Gamma_2|\alpha'\rangle &= |\alpha'\rangle \\ \Gamma_3\Gamma_4|\alpha'\rangle &= |\alpha'\rangle & \Gamma_1\Gamma_3\Gamma_4|\alpha'\rangle &= |\alpha'\rangle \end{aligned} \quad (114)$$

from which we can appreciate that  $\Gamma_2\Gamma_3\Gamma_4|\alpha'\rangle = |\alpha'\rangle$  is an extra projection condition that can be added for free, thus not breaking further supersymmetry. This proves our claim concerning this particular non-threshold  $\nu = \frac{1}{8}$  BPS state. There should be other SO(32) transformations, responsible for the same phenomena, for more involved and less supersymmetric non-threshold bound states.

We would like to finish with a brief remark concerning factorizable states at threshold. Assume their mutually commuting subsystems are SO(32) related with subsystems at threshold through  $U_i \in \text{SO}(32)$  transformations, then the full system is SO(32) related with a set of commuting single branes through the automorphism

$$U = \prod_{i=1}^n U_i \quad (n \leq 5).$$

The essential ingredient of the proof is to consider two commuting projectors  $\mathcal{P}_1, \mathcal{P}_2$  such that

$$\begin{aligned} \mathcal{P}_1 &= U_1 \Gamma_1 U_1^t \\ \mathcal{P}_2 &= U_2 \Gamma_2 U_2^t, \end{aligned} \quad (115)$$

where  $\Gamma_1^2 = \Gamma_2^2 = \mathbb{I}$ ,  $\text{tr} \Gamma_1 = \text{tr} \Gamma_2 = 0$ . By hypothesis, not only  $[\mathcal{P}_1, \mathcal{P}_2]$  vanishes but also any commutator of any single projectors in  $\mathcal{P}_1$  with any single projector in  $\mathcal{P}_2$ . Under these circumstances,  $[\Gamma_1, U_2] = [\mathcal{P}_2, U_1] = 0$  which are sufficient conditions to prove that the over-all SO(32) automorphism relating the state  $\mathcal{P}_1|\alpha\rangle = \mathcal{P}_2|\alpha\rangle = |\alpha\rangle$  to

$$\Gamma_1|\alpha'\rangle = \Gamma_2|\alpha'\rangle = |\alpha'\rangle$$

where  $|\alpha\rangle = U|\alpha'\rangle$  with  $U = U_1 U_2$ . The extension to a set of  $n$  ( $n \leq 5$ ) commuting projectors  $\{\mathcal{P}_i\}$  is straightforward.

## VI. DISCUSSION

As happens with R-symmetry in supersymmetric field theories [16], the automorphism group of a given superalgebra may or may not be a good symmetry of the theory. If it is, it may be violated by anomalies or spontaneously broken, or it may remain a good symmetry of the theory. It remains an open question to know whether the

group  $GL(32, \mathbb{R})$ , or a subgroup of it, is a symmetry of M-theory and if so, which formulation would make it manifest. In the following, we shall not try to answer these questions but we shall point out some remarks concerning the possible realization of automorphisms on world volume effective actions.

It is well-known that kappa invariant world volume brane theories do provide us with field theory realizations of the corresponding spacetime supersymmetry algebras. BPS states are realized in brane theory through field configurations solving the kappa symmetry preserving condition [17]

$$\Gamma_\kappa \epsilon = \epsilon,$$

and saturating the bound on the energy [18].

Thus, branes propagating in  $\mathcal{N} = 1$   $D = 11$  Super-Poincaré give us field theory realizations of the corresponding SuperPoincaré algebra, or truncations of it. It is then natural to wonder whether the group of automorphisms of such an algebra is a symmetry of the corresponding world volume theory. Since the Lorentz group in eleven dimensions can be seen as a subgroup of  $GL(32, \mathbb{R})$ , it is obvious that such subgroup will be linearly realized on the brane (before any gauge fixing).

In the previous section, we showed that central charges  $\mathcal{Z}$ 's are generically "rotated" among themselves under automorphism transformations. Since for bosonic configurations, such topological charges are given by world space integrals involving derivatives of the brane dynamical fields, one should also expect, if any, the existence of non-local transformations leaving certain brane theories invariant. This is the case for the non-local transformations leaving the D3-brane effective action invariant [19]. The latter are the world volume realization of the S-duality automorphism for the  $\mathcal{N} = 2$   $D = 10$  IIB SuperPoincaré algebra. The analysis performed in [20] shows that there should be similar non-local transformations giving rise to symmetry transformations of other D-brane effective actions by T-duality. We hope to come to these issues in the future.

#### Acknowledgements

JM and JS are supported by a fellowship from Comissionat per a Universitats i Recerca de la Generalitat de Catalunya. This work was supported in part by AEN98-0431 (CICYT), GC 1998SGR (CIRIT).

---

[1] C. Hull and P. K. Townsend, Nucl. Phys. **B438** (1995) 109 (hep-th/9505073);  
E. Witten, Nucl. Phys. **B443** (1995) 85 (hep-th/9503124).

[2] A. Strominger and C. Vafa, Phys. Lett. **B379** (1996) 99 (hep-th/9601029);  
C. Callan and J. M. Maldacena, Nucl. Phys. **B472** (1996) 591 (hep-th/9602043);  
J. C. Breckenridge, R. C. Myers, A. W. Peet and C. Vafa, Phys. Lett. **B391** (1997) 93 (hep-th/9602065);  
J. M. Maldacena and A. Strominger, Phys. Rev. Lett. **77** (1996) 428 (hep-th/9603060);  
C. V. Johnson, R. R. Khuri and R. C. Myers, Phys. Lett. **B378** (1996) 78 (hep-th/9603061).  
[3] G. Papadopoulos and P. K. Townsend, Phys. Lett. **B380** (1996) 273 (hep-th/9603087);  
A. A. Tseytlin, Nucl. Phys. **B475** (1996) 149 (hep-th/9604035);  
I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. **B475** (1996) 179 (hep-th/9604166);  
V. Balasubramanian and F. Larsen, Nucl. Phys. **B478** (1996) 199 (hep-th/9604189);  
J. P. Gauntlett, D. A. Kastor and J. Traschen, Nucl. Phys. **B478** (1996) 544 (hep-th/9604179);  
J. X. Lu and S. Roy, JHEP9908 (1999) 002 (hep-th/9904112); Nucl. Phys. **B560** (1999) 181 (hep-th/9904129); JHEP0001 (2000) 034 (hep-th/9905014);  
Phys. Rev. **D60** (1999) 126002 (hep-th/9905056).  
[4] E. Bergshoeff, M. de Roo, E. Eyras, B. Janssen and J. P. van der Schaar, Nucl. Phys. **B494** (1997) 119 (hep-th/9612095); Class. Quant. Grav. **14** (1997) 2757 (hep-th/9704120).  
[5] J. G. Russo and A. A. Tseytlin, Nucl. Phys. **B490** (1997) 121 (hep-th/9611047);  
M. S. Costa and G. Papadopoulos, Nucl. Phys. **B510** (1998) 217 (hep-th/9612204);  
M. S. Costa and M. Cvetič, Phys. Rev. **D56** (1997) 4834 (hep-th/9703204).  
[6] P. K. Townsend, *M-theory from its superalgebra*, in 'Strings, branes and dualities', Cargèse 1997, ed. L. Baulieu et al., Kluwer Academic Publ. 1999, p.141 (hep-th/9712004).  
[7] J. P. Gauntlett and C. M. Hull, JHEP 0001(2000) 004 (hep-th/9909098);  
I. Bando and J. Lukierski, Mod. Phys. Lett. **A14** (1999) 1257.  
[8] O. Baerwald and P. West, Phys. Lett. **B476** (2000) 157 (hep-th/9912226).  
[9] P. West, *Automorphisms, Non-Linear Realizations and Branes*, hep-th/0001216;  
P. West, *Hidden Superconformal symmetry in M theory*, hep-th/0005270.  
[10] P. K. Townsend, *p-brane democracy*, in *Particles, Strings and Cosmology*, eds. J. Bagger, G. Domokos, A. Falk and S. Kovesi-Domokos (World Scientific 1996), pp. 271-285, (hep-th/9507048).  
[11] J. W. Holten and A. Van Proeyen, J. Phys. A: Math Gen. **15** (1982) 3763.  
[12] J. P. Gauntlett, G. W. Gibbons, C. M. Hull and P. K. Townsend, *BPS states of D = 4 N = 1 supersymmetry*, hep-th/0001024.  
[13] N. A. Obers and B. Pioline, Phys. Rep. 318 (1999) 113 (hep-th/9809039).  
[14] M. Berkooz, M. R. Douglas and R. G. Leigh, Nucl. Phys. **B480** (1996) 265 (hep-th/9606139);

- N. Ohta and P. K. Townsend, Phys. Lett. **B418** (1998) 77 (hep-th/9710129) ;  
 B. S. Acharya, J. M. Figueroa-O'Farrill and B. Spence, JHEP 9804(1998) 012 (hep-th/9803260); JHEP 9807(1998) 004 (hep-th/9805073);  
 B.S. Acharya, J.M. Figueroa-O'Farrill, B.Spence and S. Stanciu, JHEP 9807(1998) 005 (hep-th/9805176).  
 [15] J.M. Izquierdo, N.D. Lambert, G. Papadopoulos and P.K. Townsend, Nucl. Phys. **B460** (1996) 560 (hep-th/9508177).  
 [16] S. Weinberg, *The quantum theory of fields. Volume III Supersymmetry*, Cambridge University Press (2000).  
 [17] E. Bergshoeff, R. Kallosh, T. Ortín and G. Papadopoulos, Nucl. Phys. **B502** (1997) 149 (hep-th/9705040).  
 [18] J. Gauntlett, J. Gomis and P. K. Townsend, JHEP9801 (1998) 003 (hep-th/9711205).  
 [19] Y. Igarashi, K. Itoh and K. Kamimura, Nucl. Phys. **B536** (1998) 469 (hep-th/9806161).  
 [20] J. Simón, Phys. Rev. **D61** 047702 (2000) (hep-th/9812095) ;  
 K. Kamimura and J. Simón, Nucl. Phys. **B585** (2000) 213 (hep-th/0003211).

Brane	Projector	Charge
Mw	$\Gamma_{0m_1}$	$\mathcal{Z}^{m_1}$
M2	$\Gamma_{0m_1 m_2}$	$\mathcal{Z}^{m_1 m_2}$
M5	$\Gamma_{0m_1 \dots m_5}$	$\mathcal{Z}^{m_1 \dots m_5}$
Mkk	$\Gamma_{0m_1 \dots m_6}$	$\mathcal{Z}^{0m_7 \dots m_{\sharp}}$
M9	$\Gamma_{0m_1 \dots m_9}$	$\mathcal{Z}^{0m_{\sharp}}$

TABLE I. Single branes preserving  $\nu = \frac{1}{2}$ , their supersymmetry projection condition and their charges.

$\perp$	M2	M5	Mkk	M9
Mw	1	1	1	1
M2	0	1	0,2	1
M5	1	1,3	1,3,5	5
Mkk	0,2	1,3,5	2,4	5

TABLE II. Threshold bound states involving two single branes.

$\perp$	Mw	M2	M5	Mkk	M9
Mw	0	0	0	0	0
M2	0	1	0,2	1	2
M5	0	0,2	0,2,4	2,4	4
Mkk	0	1	2,4	3,5	6
M9	0	2	4	6	8

TABLE III. Non-threshold bound states involving two single branes.